The symplectic 2-form for gravity in terms of free null initial data

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Abstract

A hypersurface $\mathcal{N}$ formed of two null sheets, or "light fronts", swept out by the future null normal geodesics emerging from a common spacelike 2-disk can serve as a Cauchy surface for a region of spacetime. Already in the 1960s free (unconstrained) initial data for general relativity were found for such hypersurfaces. Here an expression is obtained for the symplectic 2-form of vacuum general relativity in terms of such free data. This can be done, even though variations of the geometry do not in general preserve the nullness of the initial hypersurface, because of the diffeomorphism gauge invariance of general relativity. The present expression for the symplectic 2-form has been used previously [Rei08] to calculate the Poisson brackets of the free data.

1 Introduction

Free (unconstrained) initial data for General Relativity (GR) on certain piecewise null hypersurfaces have been known since the 1960s [Sac62, Dau63, Pen63]. In the present work the symplectic 2-form corresponding to the Einstein-Hilbert action for vacuum GR is expressed in terms of such free data on a so called *double null sheet*, a compact hypersurface $\mathcal{N}$, consisting of two null branches, $\mathcal{N}_L$ and $\mathcal{N}_R$, that meet on a spacelike 2-disk S_0 as shown in Fig. 1.[1] $\mathcal{N}_L$ and $\mathcal{N}_R$ are swept out by the two congruences of future null normal geodesics (called *generators*) emerging from S_0, and are truncated on disks S_L and S_R respectively before the generators form caustics.[2] With this symplectic 2-form the space of

[1] Some of this work has been reported in the e-print [Rei07] and in the letter [Rei08], where the symplectic 2-form is used to obtain a Poisson bracket on the free null initial data.

[2] Caustic points are points where the generators "focus"; Roughly speaking, where neighboring generators meet. More precisely, they are points where the differences in the coordinates of points of equal parameter value on neighboring generators vanishes to first order in the differences of coordinates of the base points of the two generators at S_0. This does not quite imply that the generators actually meet.

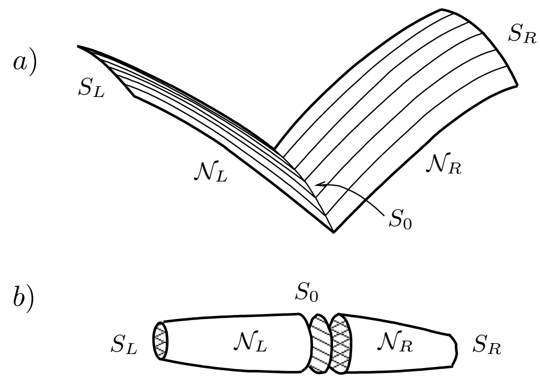

Figure 1: a) A double null sheet in 2+1 dimensional spacetime. b) In 3+1 dimensional spacetime $\mathcal{N}$ is a 3-manifold consisting of two solid cylinders joined on a disk (shown here without regard to their embedding in spacetime).

valuations of the free data becomes a phase space, which, among other things, may serve as a starting point for quantization.

In most initial value formulations of GR the initial data is subject to constraints, which complicates canonical formulations based on those data. In fact, at present the handling of the constraints absorbs most of the effort invested in canonical approches to quantum gravity. A canonical formulation based on free initial data is thus of considerable interest.

To be sure, null data is not the only way to obtain a constraint free canonical theory. York [York72] has identified spacelike free initial data, and has set up a canonical theory on spacelike hypersurfaces of uniform mean extrinsic curvature in which the most difficult constraint, the scalar constraint, has been eliminated (see [CBY80]).

A canonical framework based on null hypersurfaces is, however, especially suited for addressing certain issues. In particular the canonical framework obtained here and in [Rei08] seems ideal for attempting a semi-classical proof of Bousso's formulation of the holographic entropy bound [Bec73, tHoo93, Sus95, Bou99] in the vacuum gravity case, since a branch $\mathcal{N}_A$ ($A = L$ or R) of $\mathcal{N}$ is a "light sheet" in the terminology of Bousso [Bou99] (provided the generators are not expanding at S_0). It also seems a good classical starting point for a search for a quantization of GR respecting this entropy bound. That is, a quantization in which the area of S_0 has a discrete spectrum and each eigensubspace is of finite dimension, bounded by the exponential of the maximal entropy, according to Bousso's bound, of $\mathcal{N}_L$ and $\mathcal{N}_R$ together, i.e. by $exp(\text{Area}(S_0)/2\text{Planck area})$.[3]

[3] In order that Bousso's bound apply to both branches of $\mathcal{N}$ the generators on both sides of S_0 must be non-expanding. (This does not imply that S_0 lies in a black hole, for S_0 is a disk, not a boundaryless closed surface.) Such S_0 are easily constructed even in flat spacetime: For example, take S_0 to be a portion of the intersection of two past light cones.

In [Rei08] and the preprint [Rei07] the symplectic 2-form, $\omega_{\mathcal{N}}$, was used to calculate the Poisson brackets between initial data on $\mathcal{N}$. The main aim of the present work is to provide a detailed derivation of the expression for $\omega_{\mathcal{N}}$ that was used. The symplectic 2-form at a solution metric g takes as arguments two variations δ_1 and δ_2 belonging to the space L_g of smooth solutions to the field equations linearized about g. The expression for $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ in terms of free null initial data obtained here is valid for all "admissible" δ_1 and δ_2. Admissible variations preserve the null character of the branches of $\mathcal{N}$ and some other structures associated with $\mathcal{N}$. Because of diffeomorphism gauge invariance the expression also holds in an slightly indirect way for a much larger class of variations. If $\delta_1, \delta_2 \in L_g$ and $\delta_2 g_{ab}$ vanishes in a spacetime neighborhood of $\partial\mathcal{N}$, then there exist corresponding admissible variations δ_1' and δ_2' such that $\omega_{\mathcal{N}}[\delta_1, \delta_2] = \omega_{\mathcal{N}}[\delta_1', \delta_2']$. The symplectic product $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ may therefore be expressed in terms of the variations of the free null initial data under δ_1' and δ_2'. This suffices to obtain a Poisson bracket between the initial data.

To understand this let us briefly review how the Poisson bracket is obtained in [Rei08]: On a finite dimensional phase space with non-degenerate symplectic 2-form[4] the Poisson bracket is determined by the inverse of this 2-form. In the case of initial data for general relativity on $\mathcal{N}$ subtleties arise, both because $\mathcal{N}$ has boundaries, and because the data has infinitely many degrees of freedom. In an infinite dimensional phase space a non-degenerate symplectic 2-form can fail to have an inverse because it does not map onto the whole covector space. This is the case here. The inverse of the symplectic 2-form does not define Poisson brackets between all modes of the initial data.

This lead the author to look for a new starting point. The Peierls bracket [Pei52] is an alternative expression for the Poisson bracket which does not depend directly on the symplectic 2-form. The Peierls bracket between two functionals of spacetime fields is given by a very simple expression in terms of the first order perturbations to the solutions of the field equations occasioned by adding these functionals to the action. It's simplicity, and its direct relation to the quantum commutator give it a good claim to being a more fundamental definition of the Poisson bracket than the one in terms of the symplectic 2-form. Furthermore it agrees with the latter definition when both are defined [Pei52, DeW03, Rei07].

Unfortunately the Peierls bracket between data on $\mathcal{N}$ is ambiguous, because the perturbation generated by a functional of data on a characteristic hypersurface is discontinuous precisely at the hypersurface itself. The Peierls bracket *is* well defined on so called "observables", diffeomorphism invariant functionals $F[g]$ of the metric, with smooth functional derivatives $\delta F/\delta g_{ab}$ of compact

[4] In the present work the requierment of non-degeneracy is *not* part of the definition of a symplectic 2-form. Both symplectic and presymplectic 2-forms are referred to as "symplectic 2-forms". This is convenient because whether or not the symplectic 2-form is degenerate depends on the set of variations one admits, and on a given space of variations whether it is degenerate is generally not obvious *a priori*.

support contained in the interior of the causal domain of dependence of $\mathcal{N}$.[5][6]

The approach of [Rei08, Rei07] is to look for a Poisson bracket $\{\cdot,\cdot\}_\bullet$ on initial data that reproduces the Peierls brackets between observables. In [Rei07] it is shown that to ensure this match between the $\bullet$ bracket and the Peierls bracket (in a spacetime with metric g satisfying the field equations) it is sufficient to require that

$$\delta A = \omega_{\mathcal{N}}[\{A,\cdot\}_\bullet, \delta], \tag{1}$$

for any observable A and any δ in the space L^0_g of smooth variations which satisfy the field equations linearized about g and vanish in a spacetime neighborhood of $\partial\mathcal{N}$.

When both sides of (1) are expressed in terms of the initial data on $\mathcal{N}$ it becomes a condition on the Poisson brackets of these data.(In fact this condition is nothing but a suitably weakened form of the requirement that the Poisson bracket be inverse to the symplectic 2-form.) To express (1) in terms of initial data $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ must be expressed in terms of the initial data, but only in the case that δ_2 vanishes in some neighborhood of $\partial\mathcal{N}$.

Sachs [Sac62] and Dautcourt [Dau63] showed formally that any valuation of their null initial data on $\mathcal{N}$ determines a matching solution which is unique up to diffeomorphisms. This is the basis of their claim that their data, which is equivalent to the data we will use, is free and complete. And, of course it is the basis of the program of canonical general relativity in terms of these null initial data. Because their analyses do not address convergence issues they do not give a clear indication of the domain on which the solution exists or is unique. It seems reasonable to expect that the data in fact determine a maximal Cauchy development of $\mathcal{N}$, but what has been demonstrated rigorously so far is that a solution matching the data exists and is unique in some neighborhood of S_0 in the future of $\mathcal{N}$ [Ren90]. It has not been established that there is always a development of *all* of $\mathcal{N}$.

It is therefore worth noting that the existence and uniqueness of Cauchy developments of the data is not strictly necessary for the results of the present work. The space of data, the symplectic 2-form, and the Poisson bracket on the data found in [Rei08], are all defined independently of Cauchy developments. Indeed it is possible, and perhaps fruitful, to define a phase space of initial data on just a single branch of $\mathcal{N}$, even though the data on a single branch cannot by itself define a Cauchy development.

Given that free null initial data for GR has been available for such a long time the question arises as to why a canonical framework based on such data was not developed sooner. In fact canonical GR using *constrained data* on double null sheets has been developed by several researchers [Tor85, GRS92, GS95, d'ILV06]. Also, partial results have been obtained on the Poisson brackets of free data

[5] The causal domain of dependence $D[S]$ of a set S in a Lorentzian signature spacetime is the set of all points p such that every inextendible causal curve through p intersects S. If S is a closed achronal hypersurface one expects in physical theories that initial data on S fixes the solution in $D[S]$. See [Wald84].

[6] In [Rei07] a wide class of examples of observables in this sense is constructed, which determines the spacetime geometry of the domain of dependence, at least for generic geometries.

[GR78, GS95]. In [GR78] Gambini and Restuccia give perturbation series in Newton's constant for the brackets of free data living on the bulk of $\mathcal{N}$, but no brackets for other (necessary) data that live on the intersection surface S_0. Their results are consistent with the present work and were indeed crucial for its genesis.[7] In [GS95] Goldberg and Soteriou present distinct free data on the bulk of $\mathcal{N}$, which are claimed to form a canonically conjugate pair on the basis of a machine calculation of their Dirac brackets. It would be interesting to see if they are conjugate according to the symplectic structure obtained here.

There is however a conceptual issue which seems to have discouraged many researchers from trying to develop null canonical theory. Namely the problem of generator crossings and caustics. This problem is actually much less serious than it seems.

Let us briefly examine the problem, and its solution. Although it is not relevant to the main task of the present paper, which is to evaluate the symplectic 2-form in terms of free initial data, it is relevant to the viability of the over-all program of developing a canonical formulation of GR based on these free null initial data.

The problem is the following: Suppose a double null sheet $\mathcal{N}$ is constructed in a given solution spacetime M. It can easily happen that the generators that sweep out $\mathcal{N}$ pass through a caustic and/or cross if extended far enough. See Fig. 2. Once this occurs the generators enter the chronological future of $\mathcal{N}$ (see [Wald84] Theorem 9.3.8). In fact the segments of the generators beyond caustic or crossing points enter the interior of the domain of dependence of $\mathcal{N}$. The portion of $\mathcal{N}$ composed of these segments lies in the domain of dependence of the remainder of $\mathcal{N}$. See Appendix B of [Rei07] The initial data on part of $\mathcal{N}$ will thus be determined by the solution defined by the data on the rest of $\mathcal{N}$, which constitutes a highly complex constraint on data which was supposed to be free.[8]

Thus one would apparently wish to exclude initial data corresponding to hypersurfaces containing caustics and crossings from the phase space. A condition excluding caustics is easily found, but it seems to be much more difficult to exclude non-caustic crossing points. Presumably one would have to impose some sort of non-local inequality, which would also rob the phase space of free initial data of its simplicity.

In fact this is unneccesary. Once caustics have been excluded from $\mathcal{N}$ any further crossing points can be "unidentified" because there exists an isometric covering spacetime in which the generators do not cross, formed by pulling the metric back to the normal bundle of S_0 via the exponential map. (See Appendix

[7] The brackets between the bulk data given by the author in [Rei08] were first obtained by summing the series of Gambini and Restuccia in closed form and simplifying the result by a change of variables, before being derived more systematically from the symplectic 2-form obtained here (and in [Rei07]).

[8] This argument supposes that the solution matching the data is unique on the whole domain of dependence, which has not been established. However the solutions to the linearized field equations are certainly unique on this domain, and this already precludes independent continuous variations of the data on the part of $\mathcal{N}$ lying in the interior of the domain of dependence.

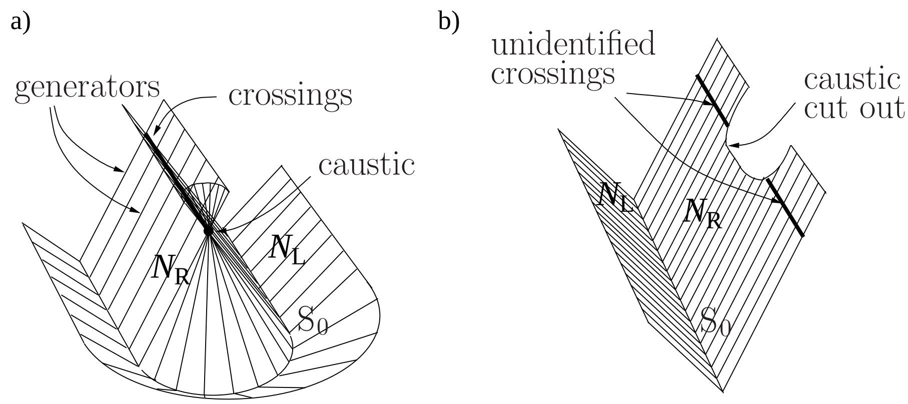

Figure 2: Panel a) shows a simple example of a caustic and intersections of generators in $2+1$ Minkowski space: S_0 is a spacelike curve having the shape of a half racetrack - a semicircle extended at each end by a tangent straight line. The congruence of null geodesics normal to S_0 and directed inward and to the future sweep out $\mathcal{N}_R$, which takes the form of a ridge roof, terminated by a half cone over the semicircle. The generators from the semicircle form a caustic at the vertex of the cone. There neighbouring generators intersect. On the other hand generators from the two straight segments of S_0 cross on a line (the ridge of the roof) starting at the caustic, but the generators that cross there are not neighbours at S_0. Clearly the generator segments beyond the crossing points enter the interior of the domain of dependence of $\mathcal{N}$. In Panel b) the double null sheet defined by S_0 in the covering space is shown, with the points that are identified in the original spacetime indicated.

B of [Rei07].) In this new spacetime no constraint forbids the independent variation of the free initial data on all parts of $\mathcal{N}$. Of course once the data is changed there is no guarantee that the spacetime regions that were unidentified in going to the covering spacetime are still isometric, so it may no longer be possible to identify them. The complicated "constraints" arising from generator crossings in the original spacetime are precisely the conditions that must be met in order that the isometry of these regions be maintained. They are *not* constraints that must be satisfied in order that a solution matching the data exists.

We are thus led to the following simple and plausible picture: Any valuation of the free data without caustics on $\mathcal{N}$ posseses a Cauchy development satisfying Einstein's equations. The Cauchy developments of a subset of valuations of the initial data, which satisfy certain complicated conditions, have isometries which allow the identification of regions so that the generators of $\mathcal{N}$ cross in the resulting spacetime.[9] Note that we have not proved that this picture is correct. That requiers a proof of the existence of solutions matching the free data throughout $\mathcal{N}$, which is not yet available. What has been shown is that the possibility of generator crossings does not represent an obstruction to this picture, nor even an argument against it.

This resolution of the problem of generator crossings suffices for the development of a simple and meaningful canonical theory based on null initial data. However, it does not mean that generator crossings are always to be regarded as unphysical. In many applications one surely would have to deal with them. But even in such cases a canonical framework based on Cauchy developments in which all generator crossings have been unidentified might provide a useful perspective.

A different conceptual issue, which *is* directly relevant to the calculation of the symplectic 2-form, is the following: The symplectic 2-form is a bilinear on perturbations of the metric satisfying the linearized field equations. Generically such perturbations do not preserve the null character of the branches of $\mathcal{N}$. How then are these perturbations to be represented by the variations of *null* initial data? The key is the diffeomorphism gauge invariance of GR. Roughly speaking, to each perturbation there corresponds a gauge equivalent one which does preserve the nullness of the branches of $\mathcal{N}$, and so can be expressed in terms of the variation of null initial data. This is only aproximately correct. As we will see, the precise resolution of the problem is rather delicate because not all diffeomorphisms are gauge in the sense of being degeneracy vectors of the symplectic 2-form.

The remainder of the paper is organized as follows: In the next section the free initial data that will be used is defined using a convenient chart on each of the branches of $\mathcal{N}$. This data is shown to be equivalent to Sachs's data, and

[9] It is worth noting that the same issue arises in the spacelike Cauchy problem, and is resolved in the same way. Spacelike hypersurfaces that enter the interior of their own domains of dependence are easily constructed in any solution spacetime M. But the unique maximal Cauchy development of the initial data induced from M on such a hypersurface is a covering manifold of the original domain of dependence, in which the hypersurface is achronal.

thus free and complete to the extent that Sachs's is. In section 3 the symplectic 2-form corresponding to the Einstein-Hilbert action is evaluated on an arbitrary hypersurface in terms of the 4-metric and its variations. A large class of infinitesimal diffeomorphisms is shown to be gauge in subsection 3.1. Section 4 is dedicated to expressing the symplectic 2-form in terms of the null initial data. In subsection 4.1 it is shown how the diffeomorphism gauge invariance can be exploited to express the symplectic 2-form in terms of null initial data on the variations that needed for the calculation of the Poisson bracket in [Rei08]. Subsection 4.2 is a discussion of the role of the diffeomorphism data. In subsection 4.3 some important charts are defined. In subsection 4.4 the symplectic potential is expressed in terms of our free the null initial data. Finally, in subsection 4.5 the symplectic 2-form is obtained in terms of these data. An appendix treats variations in fixed and moving charts.

2 The free data

2.1 coordinates on $\mathcal{N}$

A special chart $(v^A, \theta^1, \theta^2)$ will be used on each branch $\mathcal{N}_A$ ($A = L$ or R) of $\mathcal{N}$. v^A is a parameter along the generators and θ^p ($p = 1, 2$) is constant along these. Since ∂_{v^A} is tangent to the generators it is null and normal to $\mathcal{N}_A$.[10] The line element on $\mathcal{N}_A$ thus takes the form

$$ds^2 = h_{pq} d\theta^p d\theta^q, \tag{3}$$

with no dv terms. v^A is taken proportional to the square root of $\rho \equiv \sqrt{\det h}$, the area density in θ coordinates on 2D cross sections of $\mathcal{N}_A$, and normalized to 1 at S_0. Thus $\rho = \rho_0(\theta^1, \theta^2) v^2$, with ρ_0 the area density on S_0. v will be called the *area parameter*.[11]

The area parameter is related to affine parameters on the generators by the vacuum Einstein equation contracted with the tangents of the generators, $R_{vv} \equiv R[\partial_v, \partial_v] = 0$. Suppose η is an affine parameter along the generators of $\mathcal{N}_A$. Then, because of this field equation and because the generators are surface forming, the Raychaudhuri equation ([Wald84], eq. (9.2.32)) reduces to the focusing equation

$$\frac{d\theta}{d\eta} = -\frac{1}{2}\theta^2 - \sigma_{pq}\sigma^{pq}, \tag{4}$$

[10] *Proof*: Suppose t is tangent to $\mathcal{N}_A$ at $p \in \mathcal{N}_A$, then t may be Lie dragged to S_0 along $n_A \equiv \partial_{v^A}$, staying always tangent to $\mathcal{N}_A$, and

$$\partial_{v^A}(n_A \cdot t) = [\nabla_{n_A} t] \cdot n_A + [\nabla_{n_A} n_A] \cdot t. \tag{2}$$

The first term vanishes since $[\nabla_{n_A} t] \cdot n_A = [\nabla_t n_A] \cdot n_A = 1/2 \nabla_t n_A^2 = 0$. Because the generators are geodesics the second term reduces to $\alpha n_A \cdot t$ with α a scalar measuring the non-affineness (i.e. acceleration) of the parameter v^A. $n_A \cdot t$ vanishes at S_0, since there t is a sum of tangents to S_0 and n_A, which are both normal to n_A, the null normal to S_0. (2) then shows that it vanishes also at p. $\blacksquare$

[11] The index A specifying the branch $\mathcal{N}_A$ of $\mathcal{N}$ will often be dropped when there is no risk of confusion.

where θ is the expansion, and σ is the shear. Now (see [Wald84] eq. (9.2.28))

$$h_{pq}\,\theta + 2\sigma_{pq} = \pounds_k h_{pq} = \partial_\eta h_{pq} = \rho\,\partial_\eta e_{pq} + h_{pq}\,\partial_\eta \ln\rho, \tag{5}$$

where $k = \partial_\eta$ is the η tangent to the generators, and the partial derivatives are evaluated in the chart $(\eta, \theta^1, \theta^2)$. It follows that the expansion is $\theta = \partial_\eta \ln\rho = 2\partial_\eta \ln v$, that the shear is $\sigma_{pq} = \rho/2\,\partial_\eta e_{pq}$, and that

$$\sigma^{pq} = h^{ps}h^{qt}\sigma_{st} = \frac{1}{\rho^2}e^{ps}e^{qt}\sigma_{st} = -\frac{1}{2\rho}\partial_\eta e^{pq}. \tag{6}$$

Substituting these expressions into (4) one finds

$$\partial_\eta^2 \ln v + (\partial_\eta \ln v)^2 = \frac{1}{8}\partial_\eta e_{pq}\partial_\eta e^{pq}. \tag{7}$$

Finally, changing the variable of differentiation to v we obtain

$$\frac{d}{dv}\ln\left|\frac{d\eta}{dv}\right| = -\frac{v}{8}\partial_v e_{pq}\partial_v e^{pq}. \tag{8}$$

This is the key equation that relates our free initial data to Sachs' [Sac62] free initial data.

Using v as a coordinate makes avoiding caustics easy. At caustic points $v^2 \equiv \rho/\rho_0$ vanishes, so the caustic *free* $\mathcal{N}$ are represented by initial data on coordinate domains in which $v > 0$.

On the other hand, v is not always a good parameter on the generators. For instance it fails in the important special case in which $\mathcal{N}_A$ is a null hyperplane in Minkowski space, because the generators neither converge nor diverge, resulting in a v that is constant on each generator. Nevertheless, for generic $\mathcal{N}$ in generic spacetimes v is good enough. Indeed in the case of greatest interest from the point of view of the holographic entropy bound, in which the generators are converging everywhere on S_0 (v decreasing away from S_0), the focusing equation (8) ensures that v continues to decrease until a caustic is reached. Since the generator segments in $\mathcal{N}$ are truncated before reaching a caustic this implies that v is a good parameter on $\mathcal{N}$.

The area parameter v is also a good parameter if the generators are diverging at S_0, provided they are truncated before they begin to reconverge. If the generators converge on some parts of S_0 and diverge on others our methods may still be used. Suppose p is a point on S_0 at which the expansion of both the R and the L future null normals is non-zero (and suppose both the spacetime geometry and S_0 are smooth[12]), then this will also be true throughout a small disk $S_0' \subset S_0$ about p. The chart (v^A, θ^q) is thus good on each branch of a double null sheet $\mathcal{N}' \subset \mathcal{N}$ swept out by the generators emerging from S_0'.

[12] A smooth function on a domain with boundary is *defined* to be one that posses a smooth extension to an open domain. See [AMR03] chapter 7. Consequently a smooth manifold with boundary necessarily has an extension to a smooth manifold without boundary, and an embedding of a manifold with boundary is smooth iff there exists a smooth extension of the embedding to a manifold without boundary

The symplectic 2-form may thus be computed on $\mathcal{N}'$, and from it the Poisson brackets between the data on $\mathcal{N}'$. Causality requires that these in fact be all the non-zero Poisson bracket of the data on the generators through p. The only points of $\mathcal{N}$ that are causally connected to a point on these generators are the points of these generators themselves, all others are "spacelike separated" from them (see appendix B of [Rei07]), so data on distinct generators should have vanishing Poisson brackets. Indeed this is what is found when the brackets are computed [Rei08].[13]

In the following we shall assume, without great loss of generality according to the preceeding arguments, that v is a good parameter throughout each branch of $\mathcal{N}$.

Ultimately, in order to define a phase space of the gravitational field in terms of initial data we have to express all limitations on admissible solutions and coordinates as restrictions on the initial data (expressed as functions of the coordinates). Points at which the parameter v is stationary, and thus not a good parameter, turn out to be detectable in the field e_{pq} on $\mathcal{N}$, which will be one of our data. Integration of (8) yields

$$\frac{dv}{d\eta}(v) = \frac{dv}{d\eta}(v_0) \exp \int_{v_0}^{v} \frac{v}{8} \partial_v e^{pq} \partial_v e_{pq} dv. \tag{9}$$

Since $d/d\eta$ is the parallel transport of a non-zero vector at S_0, it is non-zero everywhere on the generator, so (9) implies that

$$dv|_v = dv|_{v_0} \exp \int_{v_0}^{v} \frac{v}{8} \partial_v e^{pq} \partial_v e_{pq} dv, \tag{10}$$

along the generators. Therefore if v is a good parameter ($dv \neq 0$ on the generator) at *some* value v_0, and e_{pq} is a continuously differentiable function of v then v is a good parameter at all finite values of v. A breakdown of v as a parameter requires a (sufficiently strong) singularity in $\partial_v e_{pq}$. We shall admit only initial data that is smooth in the coordinates, so v is guaranteed to be good.

On a branch $\mathcal{N}_A$ the coordinate v^A thus ranges from 1 on S_0 to its value, $\bar{v}^A$, on S_A, $\bar{v}^A$ being a smooth function of the θ which is > 0 and $\neq 1$.

2.2 The data

Two types of data will be used: geometrical data that reflect the spacetime geometry, that is the diffeomorphism equivalence class of the metric, and diffeomorphism data which reflect the choice of metric within the diffeomorphism equivalence class.

The inclusion of the diffeomorphism data may seem odd in a diffeomorphism invariant theory. However the geometrical data are not enough to express the

[13] These Poisson brackets are calculated *assuming* that v^A is a good parameter on the generators throughout $\mathcal{N}_A$. That is, they are the brackets between data on $\mathcal{N}' \subset \mathcal{N}$ and not necessarily all of $\mathcal{N}$. This is therefore not a proof that data on all distinct generators Poisson commute, but it does mean that the brackets that could be calculated are consistent with this expectation coming from causality.

symplectic 2-form on $\mathcal{N}$ for all the variations we will consider. Because $\mathcal{N}$ has a boundary, not all infinitesimal diffeomorphisms are degeneracy vectors of the symplectic 2-form, $\omega_{\mathcal{N}}$, on $\mathcal{N}$. That is, some degrees of freedom measuring diffeomorphisms of the spacetime metric are non-gauge in the sense that their variations contribute to the symplectic 2-form. In order to be able to express the symplectic 2-form in terms of the variations of initial data on $\mathcal{N}$ it is therefore necessary in general to include in the data variables parametrizing these degrees of freedom. This does not necessarily mean that the diffeomorphism data are "physical". Indeed they seem to play no essential role in the phase space formulation of vacuum general relativity within the domain of dependence of $\mathcal{N}$. They seem rather to be auxiliary quantities used in the intermediate stages of the construction of this formulation. They may however be important for the definition of quasi-local linear and angular momenta associated with $\mathcal{N}$.

The diffeomorphism data will be discussed at the end of this section. The geometrical data we will use consist of e_{pq}, specified on the branches of $\mathcal{N}$ as a function of the v and θ coordinates, and further data given only on S_0 as functions of the θ^p, namely ρ_0, $\lambda = -\ln|n_L \cdot n_R|$, and the *twist*

$$\tau_p = \frac{n_L \cdot \nabla_p n_R - n_R \cdot \nabla_p n_L}{n_L \cdot n_R}. \tag{11}$$

Here $n_A = \partial_{v^A}$ is the tangent to the generators of $\mathcal{N}_A$, and inner products $(\cdot)$ are taken with respect to the spacetime metric. These data will be called v *data*. They are *regular* if the data on S_0 are smooth functions of the θ chart, and e_{pq} is smooth in the $v\theta$ chart on each branch of $\mathcal{N}$, as well as continuous across S_0.

Smooth solutions induce regular v data on any smooth double null sheet $\mathcal{N}$, provided that on each branch the generators are either everywhere converging or everywhere diverging and free of caustics, and the θ^p form a smooth chart on S_0. (When the generators are everywhere converging or diverging, v is a smooth function without stationary points on the generators. Smooth functions on the generators are then smooth functions of v.)

Sachs [Sac62] argues that a similar set of data is free, and complete in the sense that it determines the solution geometry. Sachs' data consists of e_{pq} on $\mathcal{N}$, but given as a function of an affine parameters η on the generators instead of v, and the following data on S_0: ρ_0, $\partial_{\eta^L}\rho$, $\partial_{\eta^R}\rho$, and τ_η (which is the twist (11), but calculated from the tangents ∂_{η^A} instead of the $n_A = \partial_{v^A}$).[14]

Regular v data is equivalent to Sachs data. We will demonstrate that all regular v data determine unique corresponding Sachs data such that any solution matching the v data also matches the Sachs data, and conversely, any solution

[14] Sachs actually takes as his final datum a pair of quantities he writes as $C_{A,1} \quad A = 1, 2$. These are in fact the components of $-\tau_\eta$, as can be seen most easily from his equation 19. When a forgotten factor of 1/2 is restored and it is rewritten in our notation this equation reads

$$\frac{1}{2} C_{p,1} = \partial_{\eta^L} \cdot \nabla_p \partial_{\eta^R}. \tag{12}$$

The normalization condition $\partial_{\eta^L} \cdot \partial_{\eta^R} = -1$, which Sachs imposes on the affine parameters, implies that the right side equals $-1/2\tau_{\eta\, p}$.

matching the Sachs data matches the original v. It follows that if the Sachs data is free and complete, then regular v data is also: Suppose a solution matches a set of v data, then it also matches a uniquely determined set of Sachs data. If the Sachs data determines the solution uniquely (up to diffeomorphisms) then so does the v data. That is, the v data is complete. To establish that it is free it must be shown that any regular v data matches a solution. But if Sachs data are free then the Sachs data corresponding to the v data necessarily match a solution, and this solution also matches the v data.

In fact it has been proved by Rendall that any smooth Sachs data[15] matches a unique solution *in some neighborhood of* S_0 [Ren90], and it is a reasonable conjecture that it matches a unique solution on all of $\mathcal{N}$ provided $\mathcal{N}$ is free of caustics. (See discussion in the introduction.) The Sachs data corresponding to regular v data are indeed free of caustics on $\mathcal{N}$. Thus, if the conjecture is valid, regular v data are free and complete on $\mathcal{N}$.

We now turn to the proof of the equivalence of regular v data and Sachs data. The proof consists in demonstrating that in solution spacetimes regular v data on $\mathcal{N}$ determines the Sachs data on $\mathcal{N}$. Moreover, without assuming *a priori* that a solution matching the v data exists, Sachs data may be evaluated for *any* regular v data using the transformation that holds on solutions. Finally, it is noted that any solution matching Sachs data obtained in this way from regular v data also matches the original v data.

As already mentioned, the Sachs data differ from the v data essentially by a coordinate transformation. The Sachs data are functions of an affine parameter along the generators, while the v data are functions of the area parameter v. As the first step in the equivalence proof let us demonstrate that in a solution an affine parameter η along the generators can be calculated from the v data and the area parameter v. $\eta(v)$ then determines the map from the coordinates v, θ^1, θ^2, to which the v data are referred, to Sachs' coordinates η, θ^1, θ^2.

The field equation $R_{vv} = 0$ on $\mathcal{N}$ implies that any affine parameter η along the generators satisfies the focusing equation (8). But from the integrated form (9) of the focusing equation it is clear that e_{pq}, which is a smooth function of v on the compact interval $[1, \bar{v}]$, determines $\eta(v)$ up to an affine transformation, that is, up to a constant rescaling and a constant shift. The solutions to (9) are thus precisely the affine parameters.[16]

The shift and rescaling freedom in $\eta(v)$ can be parameterized by the values

[15] In his proof of existence and uniqueness Rendall takes as a datum $\partial_L g_{Rp}$ (where g is the 4-metric and the components are referred to the basis $d\eta^L, d\eta^R, d\theta^p$) in place of $\tau_{\eta\,p} = 2\partial_{[L} g_{R]p}$. But in Rendall's spacetime coordinates $\partial_{(L} g_{R)p}$ is determined by the remaining (Sachs) data, so his proof applies just as well if τ_η is used as the datum.

[16] When the field equation $R_{vv} = 0$ does not hold $\eta(v)$ is not an affine parameter, but it is still determined up to affine transformations, and it is a good parameter, for it follows directly from (9) that $\eta(v)$ is smooth and monotonic with non-zero derivative. If the parameter η that Sachs data is referred to is interpreted to be this parameter then all spacetime geometries, solutions or not, that match regular v data, also match the corresponding Sachs data. The only role of the field equations in the equivalence of regular v data and Sachs data is that they ensure that $\eta(v)$ is an affine parameter in accordance with the standard spacetime interpretation of Sachs data.

of η and $\partial_v \eta$ at S_0. For the Sachs coordinates η^L and η^R this amounts to four functions $A_A = \eta^A|_{S_0}$ and $B_A = \partial_{v^A} \eta^A|_{S_0}$ on S_0, which are restricted by Sachs condition $\partial_{\eta^L} \cdot \partial_{\eta^R} = -1$ at S_0. Rewriting this condition in terms of the vectors $n_A \equiv \partial_{v^A} = B_A \partial_{\eta^A}$ one obtains

$$- B_L B_R = n_L \cdot n_R = -\sigma_L \sigma_R |n_L \cdot n_R| = -\sigma_L \sigma_R e^{-\lambda}, \tag{13}$$

where $\sigma_A = 1$ if v^A increase toward the future (i.e. $\bar{v}_A > 1$), and $\sigma_A = -1$ if it decreases toward the future. (The signature of the 4-metric is taken to be $-+++$, which implies that the inner product of future directed tangents to the L and R generators is negative).

Since e^{pq} is smooth in v and v is non-stationary on the generators (9) implies that $\eta(v)$ is smooth, with non-zero derivative, and thus has a smooth inverse $v(\eta)$. As claimed, the v data (and the parameters A_L, A_R, and B_R or B_L) determine a smooth and smoothly invertible transformation from the chart (v^A, θ^p) to Sachs' chart (η^A, θ^p). The shift and rescaling degrees of freedom can be eliminated by fixing the parameters A_L, A_R, and B_R once and for all. We will set $A_L = A_R = 0$, and $B_R = 1$.

The coordinate transformation allows us to obtain e_{pq} as a function Sachs coordinates. Note that e_{pq} transforms as a scalar under this particular change of chart. This is because the line element on $\mathcal{N}_A$, $ds^2 = h_{pq} d\theta^p d\theta^q$, is degenerate, with no contribution from displacements along the generators. Since the θ coordinates are the same in the two charts the components h_{pq} at a given point on $\mathcal{N}_A$ are the same. That is, h_{pq} transforms as a scalar under the change of charts. It follows that $e_{pq} = h_{pq}/\sqrt{h}$ does also. The result of the transformation, $e_{pq}(\eta, \theta^1, \theta^2)$, is smooth, and continous across S_0.

It remains to calculate the Sachs data on S_0. Namely ρ_0, $\partial_{\eta^L} \rho$, $\partial_{\eta^R} \rho$, and τ_η. The v data of course already includes ρ_0, and the derivatives of $\rho = \rho_0 v^2$ are easily obtained from the v data:

$$\begin{aligned} \partial_{\eta^R} \rho|_{S_0} &= \rho_0 2 v^R \partial_{\eta^R} v^R|_{S_0} = 2\rho_0 B_R^{-1} = 2\rho_0, & (14) \\ \partial_{\eta^L} \rho|_{S_0} &= \rho_0 2 v^L \partial_{\eta^L} v^L|_{S_0} = 2\rho_0 \sigma_L \sigma_R e^{\lambda} B_R = 2\rho_0 \sigma_L \sigma_R e^{\lambda}. & (15) \end{aligned}$$

Finally, $\tau_{\eta\, p}$ is given by the same expression as the v datum $\tau_p = [n_L \cdot \nabla_p n_R - n_R \cdot \nabla_p n_L]/n_L \cdot n_R$, but with the vectors n_A substituted by $\partial_{\eta^A} = B_A^{-1} n_A$. Thus

$$\begin{aligned} \tau_\eta &= \tau + d \ln |B_L| - d \ln |B_R|, & (16) \\ &= \tau - d\lambda - 2 d \ln |B_R| = \tau - d\lambda. & (17) \end{aligned}$$

All the Sachs data are determined by the v data.[17] Note that even if it is not assumed that the v data matches a solution, the function $\eta(v)$, and thus the Sachs datum of $e^{pq}(\eta)$, may be calculated from any regular v data using (9). Similarly, the remaining Sachs data may be obtained, from (14), (15), (17).

[17] The sign $\sigma_L \sigma_R$ is also needed to determine the Sachs data. This sign is implicit in the specification of the v data. The datum e^{pq} on each generator is given for a range of v from 1 to $\bar{v}$, and σ_A is the sign of $\bar{v}^A - 1$.

The transformation from regular v data to Sachs data we have found is invertible. Solving (15) and (17) yields:

$$\lambda = \ln|\partial_{\eta^R}\rho|_{S_0}| + \ln|\partial_{\eta^L}\rho|_{S_0}| - 2\ln(2\rho_0) \quad (18)$$

$$\tau = \tau_\eta + d\ln|\partial_{\eta^L}\rho|_{S_0}| - d\ln|\partial_{\eta^R}\rho|_{S_0}|. \quad (19)$$

(For Sachs data corresponding to regular v data the derivatives of ρ appearing as denominators or arguments of logarithms do not vanish.)[18] The focusing equation (8) can be rewritten in the form

$$\partial_\eta\partial_\eta v = \frac{v}{8}\partial_\eta e_{pq}\partial_\eta e^{pq}. \quad (20)$$

Since $v^A = 1$ on S_0, and the Sachs data determine $\partial_{\eta^A} v^A|_{S_0} = \frac{1}{2\rho_0}\partial_{\eta^A}\rho|_{S_0}$ and $e_{pq}(\eta^A)$, (20) has a unique solution $v^A(\eta^A)$ on each branch. $v^A(\eta^A)$ and $e_{pq}(\eta^A,\theta)$ then determine the v datum $e_{pq}(v^A,\theta)$, showing that all v data can be reconstructed from the Sachs data. If the Sachs data was obtained by transforming regular v data this inverse transformation yields the original v data. But the transformation relates the Sachs data and the v data of a solution. Thus if a solution matches the Sachs corresponding to a set of regular v data then this solution must also match the original v data. This completes the demonstration of the equivalence of regular v data and Sachs data.

Let us turn to the diffeomorphism data. The diffeomorphism data that will be used are $\bar{v}_A(\theta)$, the area parameter at the endpoint on S_A of the generator specified by θ, and $s^k_A = y^k_A(\theta)$, a map which gives the position of this endpoint in a *fixed* chart y_A on S_A.[19] The status of the $\bar{v}^A$ as a datum is curious. It is implicit in the specification of the v data on $\mathcal{N}_A$, since it defines the range of v^A on which e^{qp} is given, but it is not *functions* of the v data. It is independent of the v data if e^{qp} is specified on a fixed, reference range of $v\theta$ coordinates and $\bar{v}^A$ delimits the subset of this range that corresponds to points on $\mathcal{N}$. The entire set of data, consisting of the diffeomorphism data $\bar{v}_A$ and s_A and the v data, is then free, since both diffeomorphism data can be varied independently of the v data by acting on the spacetime metric with suitable diffeomorphisms, which of course map solutions to solutions.

These data and their variations suffice to determine $\omega_{\mathcal{N}}[\delta_1,\delta_2]$, when the variations δ_1 and δ_2 are what we will call "admissible". (This is explained in detail in subsection 4.1.) This is enough for our purposes because the evaluation of the Poisson brackets of the data carried out in [Rei07] and [Rei08] requiers the the symplectic 2-form only on admissible variations.

The diffeomorphism data play a role in the calculation of the Poisson bracket in [Rei08], but are they essential? Should they be regarded as "physical"? It

[18] Sachs data obtained by setting $B_R = 1$ satisfies $\partial_{\eta^R}\rho|_{S_0} = 2\rho_0$. In general $B_R = \frac{2\rho_0}{\partial_{\eta^R}\rho|_{S_0}}$.

[19] Spacetime is modeled by a manifold, and manifolds consist of individual *a priori* identifiable points. It makes sense to compare different metrics at the same point (as is done, for example, when varying the action), and one may distinguish between charts that depend on the metric field, such as normal coordinates or our $v\theta$ chart, and fixed charts which are, so to speak, painted on the manifold. See appendix A for a more detailed discussion.

seems to depend on what one wants to do. Because the observables defined in the introduction are diffeomorphism invariant they do not depend on the diffeomorphism data.[20] The requierment that the Poisson bracket of the data reproduce the brackets between observables, which (1) ensures, can thus at most determine the brackets between the v data. In subsection 4.2 it will be shown directly that (1) does not determine the brackets of the diffeomorphism data, indeed it does not involve them at all.

It seems therefore that in the canonical theory of the gravitational field in the domain of dependence of $\mathcal{N}$ the diffeomorphism data have only an auxiliary role. Indeed, the v data are found to form a closed Poisson subalgebra, that is, their brackets are functions only of v data [Rei08], so the diffeomorphism data could be eliminated altogether from the canonical formalism.

On the other hand the diffeomorphism data may be relevant to quasi-local energy or other quantities associated with the boundary $\partial\mathcal{N}$. Note that the diffeomorphism data, unlike the remaining data, "know" about the boundary $\partial\mathcal{N}$. In the present work the diffeomorphism data are included in the initial data because the expression for the symplectic 2-form used in the calculation of the Poisson brackets in [Rei08] does depend on them, and it is the main aim of the present work to present a derivation of this expression.

Before closing this subsection let us state precisely the complete set of data to be used: It consists of

- 10 real C^∞ functions, ρ_0, λ, τ_p, $\bar{v}_A$, and s^i_A, on a domain $D \in \mathbb{R}^2$ having the topology of a closed disk, with $\bar{v}_A > 0$ and $\neq 1$
- two C^∞, real, symmetric, unimodular 2×2 matrix valued functions (e_{pq} on $\mathcal{N}_L$ and $\mathcal{N}_R$) on the domains $\{\theta \in D, \min(1, \bar{v}_A(\theta)) \leq v^A \leq \max(1, \bar{v}_A(\theta))\}$, $A = L, R$ which match at $v^L = v^R = 1$ (i.e. on S_0).

Our phase space is the space of valuations of these data.

3 The symplectic 2-form of the Einstein-Hilbert action

In the present section we will define the symplectic 2-form ω_Σ of any oriented hypersurface Σ embedded in spacetime, and calculate it in terms of the spacetime metric and its variations. In subsecuent sections this expression is reduced to one in terms of our free initial data in the special case that the hypersurface is an embedded double null sheet.

[20] Not all diffeomorphisms are gauge, that is, not all are generated by degeneracy vectors of the symplectic 2-form. Nevertheless, within any region which excludes a neighborhood of the boundary $\partial\mathcal{N}$, any diffeomorphism may be realized as the restriction of a diffeomorphism which vanishes in a neighborhood of $\partial\mathcal{N}$, and such diffeomorphisms are gauge, as will be shown in subsection 3.1. Thus on the *interior* of the domain of dependence $D[\mathcal{N}]$ of $\mathcal{N}$ all gauge invariant degrees of freedom of the metric are diffeomorphism invariant. The observables seem to completely capture these degrees of freedom, at least for metrics without isometries [Rei07].

ω_Σ will be defined for metrics g that satisfy the vacuum field equations and variations that lie in the space L_g of solutions to the field equations linearized about g. Although ω_Σ is an on shell quantity it depends on the off shell action (on solutions the Einstein-Hilbert action is zero!), and it is most naturally defined as a pullback to the space of solutions of a symplectic 2-form[21] Ω_Σ defined by the action functional on all smooth metrics and variations. More precisely, ω_Σ is the restriction of Ω_Σ to metrics that satisfy the field equations and to variations in L_g.[22] (See [LW90] for the uses of Ω_Σ.)

The symplectic 2-form will be calculated from the Einstein-Hilbert action,

$$I = \frac{1}{16\pi G}\int_Q R\varepsilon, \tag{21}$$

where ε is the metric 4-volume form and Q is the domain of integration, which may be chosen freely. The sign conventions for the curvature tensor and scalar are those of [Wald84], that is, $R = R_{ab}{}^{ab}$ with

$$[\nabla_a, \nabla_b]\beta_c = R_{abc}{}^d\beta_d \tag{22}$$

for any 1-form β.

The variation of the action due to a variation δ of the metric consists of a bulk term, which vanishes on solutions, and a boundary term which determines the symplectic 2-form. The variation of ($16\pi G$ times) the Einstein-Hilbert Lagrangean is

$$\delta[R\varepsilon] = [R_{ab} - \frac{1}{2}Rg_{ab}]\delta g^{ab}\varepsilon + \delta R_{ab}g^{ab}\varepsilon, \tag{23}$$

where $R_{ab} = R_{acb}{}^c$ is the Ricci tensor. Clearly the first term vanishes on solutions. The second term is a divergence: From the definition (22) it follows that $\delta R_{abc}{}^d = -2\nabla_{[a}\delta\Gamma^d_{b]c}$ so

$$\delta R_{ab}g^{ab}\varepsilon = -2\nabla_a\delta\Gamma^{[c}_{cb}g^{a]b}\varepsilon. \tag{24}$$

The integral of this divergence is the boundary term in the variation of the action. For any vector field v

$$\nabla_a v^a\varepsilon = d\wedge v \lrcorner\, \varepsilon, \tag{25}$$

so $\delta R_{ab}g^{ab}\varepsilon = d\wedge\alpha$ with[23]

$$\alpha = -2\delta\Gamma^{[c}_{cb}g^{a]b}\varepsilon_{a\cdots} \tag{26}$$

[21] Recall that in the present work the term symplectic 2-form subsumes degenerate forms, which are often called "presymplectic" in the literature.

[22] At linearization stable solutions this is the pullback to the space of solutions since there L_g coincides with the tangent space to the solution manifold. At non-linearization stable solutions L_g is larger than the tangent space. According to the local linearizations stability theorem of [BRS87] all solutions are linearization stable in the interior of the domain of dependence of $\mathcal{N}$. In fact, whether or not L_g coincides with the tangent space to the manifold of solutions will not affect our considerations.

[23] We will occasionally mix abstract index notation with index free notation for differential forms. In particular abstract index notation will be used to indicate contractions of tensors. To avoid confusion when some indices of a tensor are written and other, uncontracted, indices are not, the unwritten indices are indicated by dots.

The boundary term in the variation of the action is thus

$$B[\delta] = -\frac{1}{8\pi G}\int_{\partial Q} \delta\Gamma^{[c}_{cb} g^{a]b}\, \varepsilon_{a\cdots}, \tag{27}$$

The *symplectic potential* associated with a portion Σ of ∂Q is obtained by restricting the boundary integral (27) to Σ:

$$\Theta_\Sigma[\delta] = -\frac{1}{8\pi G}\int_{\Sigma} \delta\Gamma^{[c}_{cb} g^{a]b}\, \varepsilon_{a\cdots}. \tag{28}$$

The *symplectic 2-form* on a pair of variations δ_1 and δ_2 is

$$\begin{aligned} \Omega_\Sigma[\delta_1,\delta_2] &\equiv \delta_1\Theta_\Sigma[\delta_2] - \delta_2\Theta_\Sigma[\delta_1] - \Theta_\Sigma[[\delta_1,\delta_2]] && (29)\\ &= -\frac{1}{8\pi G}\int_\Sigma \delta_2\Gamma^{[c}_{cb}\delta_1(g^{a]b}\varepsilon_{a\cdots}) - (1\leftrightarrow 2) && \\ & && (30) \end{aligned}$$

See [CW87] and [LW90]. $\Omega_\Sigma[\delta_1,\delta_2]$ may be interpreted as the curl of Θ_Σ in the space of metric fields, evaluated on two tangent vectors, δ_1 and δ_2, to this space.

The definition of Θ_Σ given is in fact ambiguous. The boundary integral B in the variation is quite unambiguously defined, but the integrand of B is not. Adding an exact form to it would not affect B, but would alter Θ_Σ by an integral over $\partial\Sigma$. There is also the freedom to add a boundary term to the action. At first sight it would seem that such a boundary term only adds a total variation to Θ_Σ, which would not affect Ω_Σ. However whether this is so actually depends on the precise prescription used to determine the integrand of Θ_Σ from the Lagrangean. Lee and Wald [LW90] give such a prescription (in which boundary terms added to the action *can* produce boundary terms in ω_Σ if they depend on derivatives of the fields). Our expression (28) for Θ_Σ corresponds to the Einstein-Hilbert action without boundary term according to this prescription. But is there a physical reason to prefer the Lee-Wald prescription? Are boundary terms in ω_Σ important?

The Poisson bracket should not depend on boundary terms. The Peierls bracket is expressed directly in terms of the advanced and retarded Green's functions, which are not affected by boundary terms in the action.[24] The Poisson bracket $\{\cdot,\cdot\}_\bullet$ on initial data calculated in [Rei07] and [Rei08], which is defined by the requierment that it reproduce the Peierls bracket, should also be insensitive to boundary terms. Indeed the condition (1) which ensures the matching to the Peierls bracket is manifestly unaffected by the addition of boundary terms to

[24] The Greens functions depend only on the field equations derived from the action with a suitable source term. The boundary terms we are considering are ones like the York-Gibbons-Hawking term, which are matched to boundary conditions on the variations of the fields so that the presence of the boundary does not affect the field equations that result from extremizing the action. We are not considering boundary terms which represent a physical feature at the boundary, and of course would affect Greens functions.

the symplectic 2-form.[25] Note that brackets obtained in [Rei08] do not “know” where the boundary $\partial\mathcal{N}$ is, That is, they are unchanged by a displacement of the boundary, except in the case of the brackets of the diffeomorphism data which themselves encode features of the boundary.

On the other hand, the canonical generators of diffeomorphisms that move the boundary $\partial\Sigma$ *do* seem to depend on boundary terms in ω_Σ. Such generators define quasi-local notions of energy, angular momentum, etc. and the correct boundary terms would presumably be defined by the properties one wants these quasi-local quantities to have. This interesting direction will not be explored here. Rather we shall simply adopt the symplectic potential (28) corresponding to the Einstein-Hilbert action without boundary term.

3.1 diffeomorphisms

The degeneracy vectors of the symplectic 2-form are variations Δ such that $\omega_\Sigma[\Delta,\delta] = 0$ for all smooth solutions to the linearized field equations δ. These are often called *gauge variations* although it is not clear that this is the most apropriate definition of “gauge” when Σ has boundaries. In general relativity the degeneracy variations of the metric are Lie derivatives of the metric along vector fields satisfying certain conditions at $\partial\Sigma$. This is the familiar diffeomorphism invariance of general relativity: If ψ_t is a family of diffeomorphisms parameterized by $t \in \mathbb{R}$ then the t derivative of the image metric $\psi_t^*(g)$ is $d\psi_t^*(g)/dt = -\pounds_v g$, where v is the field of tangents to the orbits of the manifold points under ψ_t, so Lie derivatives generate diffeomorphisms.

Let us evaluate $\omega_\Sigma[\pounds_v,\delta]$ for any C^∞ vector field v on spacetime and $\delta \in L_g$.

$$\omega_\Sigma[\pounds_v,\delta] = \pounds_v\Theta_\Sigma[\delta] - \delta\Theta_\Sigma[\pounds_v] - \Theta_\Sigma[[\pounds_v,\delta]]. \tag{31}$$

Now

$$\Theta_\Sigma[\delta] = \frac{1}{16\pi G}\int_\Sigma \alpha \tag{32}$$

with α the 3-form defined in (26). Thus

$$\begin{aligned} \pounds_v\Theta_\Sigma[\delta] &= \frac{1}{16\pi G}\int_\Sigma \pounds_v\alpha & (33)\\ &= \frac{1}{16\pi G}\int_\Sigma v \lrcorner\, [d\wedge\alpha] + d\wedge[v \lrcorner\, \alpha]. & (34)\end{aligned}$$

But $d\wedge\alpha = \delta R_{ab}g^{ab}\varepsilon$ is the divergence term in the variation of the Einstein-Hilbert Lagrangean density, which vanishes because δ satisfies the linearized

[25] The condition (1) does not determine the brackets of all the data uniquely. In [Rei08] the bracket is therefore derived from a strengthened version of (1) which could be affected by boundary terms in the action. But as long as no compelling motivation is found for the auxiliary conditions used to obtain a unique bracket, any such sensitivity to boundary terms has to be regarded as artificial.

vacuum field equation $\delta R_{ab} = 0$. Therefore

$$\begin{aligned}
&\pounds_v \Theta_\Sigma[\delta] && (35)\\
&= \frac{1}{16\pi G}\int_{\partial\Sigma} v \lrcorner\, \alpha && (36)\\
&= -\frac{1}{8\pi G}\int_{\partial\Sigma} v^b \delta\Gamma^{[c}_{cd} g^{a]d}\, \varepsilon_{ab\cdot\cdot} && (37)\\
&= \frac{1}{8\pi G}\int_{\partial\Sigma} v^a (\nabla^b \delta\varepsilon_{ab\cdot\cdot} + \frac{1}{2}\nabla_c \delta g^{bc}\varepsilon_{ab\cdot\cdot}). && (38)
\end{aligned}$$

In the last line the identity

$$\delta\Gamma^c_{ab} = \frac{1}{2} g^{cd}\{\nabla_b \delta g_{da} + \nabla_a \delta g_{db} - \nabla_d \delta g_{ab}\}. \tag{39}$$

has been used.

The second term in (31) is the δ variation of

$$\Theta_\Sigma[\pounds_v] = -\frac{1}{8\pi G}\int_\Sigma \pounds_v \Gamma^{[c}_{cb} g^{a]b}\, \varepsilon_{a\cdots} \tag{40}$$

But (39) and Einstein's field equation, which g satisfies, imply that

$$\begin{aligned}
&\pounds_v \Gamma^{[c}_{cb} g^{a]b} \varepsilon_{a\cdots} \\
&= (\nabla^a \nabla_c v^c - \frac{1}{2}\nabla_c \nabla^c v^a - \frac{1}{2}\nabla_c \nabla^a v^c)\varepsilon_{a\cdots} && (41)\\
&= \nabla_c \nabla^{[a} v^{c]} \varepsilon_{a\cdots} && (42)\\
&= \frac{1}{2} d \wedge (\nabla^a v^b \varepsilon_{ab\cdot\cdot}). && (43)
\end{aligned}$$

Thus

$$\Theta_\Sigma[\pounds_v] = -\frac{1}{16\pi G}\int_{\partial\Sigma} \nabla^a v^b\, \varepsilon_{ab\cdot\cdot} \tag{44}$$

Since

$$[\pounds_v, \delta] g = -\pounds_{\delta v} g \tag{45}$$

it follows that

$$\Theta_\Sigma[[\pounds_v, \delta]] = \frac{1}{16\pi G}\int_{\partial\Sigma} \nabla^a \delta v^b\, \varepsilon_{ab\cdots}. \tag{46}$$

Subtracting (46) and the δ variation of (44) from (38) one obtains

$$\begin{aligned}
&\omega_\Sigma[\pounds_v, \delta] \\
&= \frac{1}{16\pi G}\int_{\partial\Sigma} v^a (2\nabla^b \delta\varepsilon_{ab\cdot\cdot} + \nabla_c \delta g^{bc} \varepsilon_{ab\cdot\cdot}) \\
&\qquad + \delta(\nabla^a v^b \varepsilon_{ab\cdot\cdot}) - \nabla^a \delta v^b \varepsilon_{ab\cdot\cdot}, && (47)\\
&= \frac{1}{16\pi G}\int_{\partial\Sigma} 3 v^{[a} \delta\Gamma^c_{cd} g^{b]d} \varepsilon_{ab\cdot\cdot} + \delta[g^{ca}\varepsilon_{ab\cdot\cdot}]\nabla_c v^b. && (48)
\end{aligned}$$

This integral obviously vanishes when v and ∇v vanish on $\partial\Sigma$. The corresponding variation $\pounds_v$ is therefore a degeneracy vector of the symplectic 2-form.

4 The symplectic 2-form on $\mathcal{N}$ in terms of the free null data.

In this section the symplectic 2-form $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ defined in section 3 will be expressed in terms of the free null initial data defined in subsection 2.2, for variations δ_1 and δ_2 that satisfy the linearized field equations and a series of further conditions that define what we will call the "admissible variations".

Admissible variations are fairly special, but our expression for $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ will in fact be applicable to a much larger class of variations. We will show that any pair of variations, $\delta_1 \in L_g$ and $\delta_2 \in L_g^0$, may be replaced in $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ by corresponding admissible variations without changing the value of the symplectic 2-form. Our expression for the symplectic 2-form in terms of the free null data therefore suffices to convert (1) into an explicit condition on the Poisson brackets of these data.

In [Rei08] the Poisson brackets of the initial data are obtained from a somewhat strengthened version of (1), which can also be expressed in terms of the initial data using the expression for $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ on admissible variations.

The first subsection treats conceptual issues involved in expressing $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ in terms of the null data of 2.2 and demonstrates that attention may be restricted to the class of admissible variations. The next subsection demonstrates the limited role of the diffeomorphism data defined in 2.2. The third subsection presents some charts used in the calculations. In the fourth subsection the symplectic potential is evaluated in terms of the free null data. Finally, in the last subsection, this expression for the symplectic potential is used to calculate the symplectic 2-form in terms of the free null data.

4.1 Variations in terms of null initial data and admissible variations

According to (30) the symplectic 2-form on $\mathcal{N}$, at a given spacetime metric g, is

$$\omega_{\mathcal{N}}[\delta_1, \delta_2] = -\frac{1}{8\pi G}\int_{\mathcal{N}} \delta_2 \Gamma^{[c}_{cb}\delta_1(g^{a]b}\varepsilon_{a\cdots}) - (1 \leftrightarrow 2), \tag{49}$$

where $\delta_1 g$ and $\delta_2 g$ are solutions to the field equations linearized about g. Our task is to express $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ in terms of the free null initial data and their variations in the case that $\mathcal{N}$ is a double null sheet of g, $\delta_1 \in L_g$, and $\delta_2 \in L_g^0$, the set of solutions to the linearized field equations that vanish in a spacetime neighborhood of $\partial\mathcal{N}$.

It is not *a priori* obvious that this can be done. By definition the spacetime metrics matching the null data make the hypersurfaces $\mathcal{N}_L$ and $\mathcal{N}_R$ null, so the variations of these data only parametrize variations $\delta \in L_g$ that preserve the nullness of $\mathcal{N}_L$ and $\mathcal{N}_R$. [26] Arbitrary variations will not in general do this

[26] It *is* possible to define variations of null data under general variations of the metric, if the null data live not on $\mathcal{N}$ but on a metric dependent double null sheet associated with $\mathcal{N}$. Working along these lines one arrives ultimately at the same theory presented here.

for a given, fixed, hypersurface $\mathcal{N}$. That the symplectic 2-form $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ can nevertheless be expressed in terms of the variations of null data for all $\delta_1 \in L_g$ and $\delta_2 \in L_g^0$ is a consequence of the diffeomorphism gauge invariance of general relativity.

Although the branches of the fixed hypersurface $\mathcal{N}$ may cease to be null when the spacetime metric is changed slightly, it is always possible, by a small deformation of $\mathcal{N}$, to obtain a new hypersurface $\mathcal{N}'$ which is a double null sheet of the new metric. (The double null sheet $\tilde{\mathcal{N}}_{S_0}$ swept out by the future null normal geodesics from S_0 in the new metric is an example.) Thus, if the given change in the metric is followed by the action on the metric of a suitable diffeomorphism, which moves $\mathcal{N}'$ to $\mathcal{N}$, then the resulting total alteration of the metric preserves the double null sheet character of $\mathcal{N}$. Any variation δ may therefore be split into the sum of a null sheet preserving variation δ', that is, one that preserves the null sheet character of $\mathcal{N}$, and a diffeomorphism generator $\pounds_u$.

Applying this decomposition to the two arguments $\delta_1, \delta_2 \in L_g$ of the symplectic 2-form one obtains

$$\omega_{\mathcal{N}}[\delta_1, \delta_2] = \omega_{\mathcal{N}}[\delta_1', \delta_2'] + \omega_{\mathcal{N}}[\delta_1', \pounds_{u_2}] + \omega_{\mathcal{N}}[\pounds_{u_1}, \delta_2'] + \omega_{\mathcal{N}}[\pounds_{u_1}, \pounds_{u_2}]. \tag{50}$$

If the diffeomorphism generators are degeneracy vectors of $\omega_{\mathcal{N}}$, that is, if they are gauge, then all terms but the first vanish, and in this first term only the null sheet preserving variations δ' appear. In this case, the fact that the variations of null data can only parametrize nullness preserving variations would not be an impediment to expressing the symplectic 2-form in terms of these data. Indeed, the v data of subsection 2.2 determines the metric and its first derivatives on $\mathcal{N}$ up to diffeomorphisms that map $\mathcal{N}$ to itself. If the generators of all such diffeomorphisms were degeneracy vectors then the v data and their variations would suffice by themselves to determine $\omega_{\mathcal{N}}[\delta_1', \delta_2']$; The v data and their variations would determine the metric and its derivatives, and the gauge equivalence class of their variations under δ_1' and δ_2', up to a diffeomorphism mapping $\mathcal{N}$ to itself, and the integral (49) is invariant under such diffeomorphisms.

However, not all diffeomorphism generators are degeneracy vectors of the symplectic 2-form. Eq (48) shows that the diffeomorphism terms in (50) are integrals over the boundary of $\mathcal{N}$ that might not vanish. Indeed some $\delta \in L_g$ might not be gauge equivalent to *any* null sheet preserving variation. That is, there might exist no null sheet preserving variation δ' such that $\delta' - \delta$ is a degeneracy vector. (In fact it seems plausible that this is the case for some δ, but it has not been demonstrated.) If this is so then $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ cannot be expressed in terms of null initial data for all $\delta_1, \delta_2 \in L_g$.

Fortunately we do not need to express $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ in terms of null initial data for completely general δ_1 and δ_2 in L_g. We are interested in the case in which δ_1 is arbitary but $\delta_2 g_{ab}$ vanishes in a spacetime neighborhood of $\partial\mathcal{N}$,[27] that is,

[27] It seems that the same results can be obtained with the weaker condition that $\delta_2 g_{ab}$ and $\delta_2 \nabla_c g_{ab}$ vanish on $\partial\mathcal{N}$ itself. We will not pursue this matter here.

$\delta_1 \in L_g$, $\delta_2 \in L_g^0$, because this is the case relevant for the calculation of the Poisson bracket via (1) in [Rei08].

Let us suppose then that $\delta_2 \in L_g^0$. This restriction implies that the diffeomorphism terms in (50) do vanish: If δ_2 vanishes in a neighborhood of $\partial\mathcal{N}$ then $\omega_{\mathcal{N}}[\pounds_{u_1}, \delta_2]$ is zero because it is an integral over $\partial\mathcal{N}$ of an integrand proportional to $\delta_2 g_{ab}$ and its derivatives there. Furthermore, when δ_2 vanishes in a neighborhhood of $\partial\mathcal{N}$ the field u_2 may be chosen so that it also vanishes in a (generally different) neighborhood of $\partial\mathcal{N}$ (see below). This implies that $\omega_{\mathcal{N}}[\delta_1', \pounds_{u_2}]$ is also zero. Thus, when $\delta_2 \in L_g^0$

$$\omega_{\mathcal{N}}[\delta_1, \delta_2] = \omega_{\mathcal{N}}[\delta_1', \delta_2']. \tag{51}$$

This means that $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ can be expressed in terms of null initial data, that is, in terms of data sufficient to determine the metric and its derivatives up to gauge on $\mathcal{N}$ assuming $\mathcal{N}$ is a double null sheet. In section 4.5 such an expression is given explicitly, in terms of the free null data defined in subsection 2.2.

Before continuing let us return to the diffeomorphism generator $\pounds_{u_2} = \delta_2 - \delta_2'$ and show that u_2 may indeed be chosen so that it vanishes in a neighborhood of $\partial\mathcal{N}$. To this end we define a new metric dependent double null sheet $\tilde{\mathcal{N}}_{\partial\mathcal{N}}$ swept out by past normal null geodesics from S_L and S_R rather than future normal null geodesics from S_0: The generators of $\mathcal{N}$ may be regarded as normal null geodesics emerging to the past from S_L or S_R and truncated where they meet at S_0. When the metric is changed these past normal null geodesics from S_L and S_R are also changed, and sweep out new null hypersurfaces $\tilde{\mathcal{N}}_{S_L}$ and $\tilde{\mathcal{N}}_{S_R}$. If δ is a variation that vanishes in a neighborhood $\mathcal{W}$ of $\partial\mathcal{N}$ then it will not disturb the geodesics that make up the portion $\partial\mathcal{N} - S_L - S_R$ of the boundary of $\mathcal{N}$, and these will still meet at ∂S_0. Furthermore, if the change in the metric is small enough $\tilde{\mathcal{N}}_{S_L}$ and $\tilde{\mathcal{N}}_{S_R}$ will intersect on a disk, $\tilde{S}_0$, where they may be truncated, and thus truncated will contain no caustics. Thus $\tilde{\mathcal{N}}_{\partial\mathcal{N}} = \tilde{\mathcal{N}}_{S_L} \cup \tilde{\mathcal{N}}_{S_R}$ is a double null sheet of the perturbed metric. It is clear that the perturbed and unperturbed generators from S_A coincide until they leave $\mathcal{W}$. Thus $\tilde{\mathcal{N}}_{\partial\mathcal{N}}$ coincides with $\mathcal{N}$ in a neighborhood of S_A, and also in a neighborhood of $\partial\mathcal{N} - S_L - S_R$, since generators sufficiently near $\partial\mathcal{N} - S_L - S_R$ never leave $\mathcal{W}$. (This follows from the compactness of the generator segments that sweep out $\partial\mathcal{N} - S_L - S_R$.)[28] As a consequence $\tilde{\mathcal{N}}_{\partial\mathcal{N}}$ can be mapped to $\mathcal{N}$

[28] The branch $\mathcal{N}_A$ of $\mathcal{N}$ is the image under the exponential map of a compact solid cylinder N_A in the normal bundle of S_A, the generators being the images of parallel straight null lines in N_A which will also be called generators. The preimage $Z \subset N_A$ of the subset $\mathcal{W} \cap \mathcal{N_A}$ of $\mathcal{N}_A$ on which the metric is invariant is open in N_A, since $\mathcal{W} \cap \mathcal{N}_A$ is open in $\mathcal{N}_A$ and the exponential map is continuous. (Here the open sets in a subset S of an ambient space X are the intersections of open sets of X space with the subset S.) Thus Z can be expressed as a union of open solid cylinders of the form $c = l \times x$, with l an open line segment parallel to the generators and $x \subset S_A$ is in S_A. Since any generator from ∂S_A lies in Z it is covered by these cylinders. But since it is compact it has a finite subcover c_i. The intersection $y = \cap_i x_i$ is a neighborhood of the base point of the generator in ∂S_A, open in S_A, such that generators from y lie entirely in Z. Taking the union of such ys one obtains an open neighborhood Y of ∂S_A in S_A such that all generators from Y remain in Z until they leave N_A. See [Rei07] proposition B.8. for a different proof.

by a diffeomorphism that reduces to the identity in a neighborhood of $\partial\mathcal{N}$. That is, $\delta_2 \in L_g^0$ implies that u_2 may be chosen to vanish in such a neighborhood.

Our fundamental condition defining the Poisson bracket (1), $\delta A = \omega_{\mathcal{N}}[\{A,\cdot\}_\bullet, \delta]\ \forall \delta \in L_g^0$, may also be expressed in terms of the null initial data. The variation $\{A,\cdot\}_\bullet$ is already null sheet preserving by virtue of its definition: The bracket $\{\cdot,\cdot\}_\bullet$ is a Poisson bracket on the null initial data, so $\{A,\cdot\}_\bullet$ is a variation of these initial data, which of course defines a null sheet preserving variation of the spacetime metric (up to diffeomorphisms which do not affect the value of $\omega_{\mathcal{N}}[\{A,\cdot\}_\bullet, \delta]$ when $\delta \in L_g^0$). (See [Rei07] appendix C.) The variations δ may be restricted to null sheet preserving variations without weakening the condition on $\{A,\cdot\}_\bullet$ that (1) implies: The variation δ may be replaced by δ' on both sides of the equation without altering the value of either, on the left because the observable A is diffeomorphism invariant, and on the right because of (51). Finally, any variation of A may be written as a sum of the corresponding variations of the initial data integrated against suitable smearing functions. The smearing functions are the functional derivatives of A by the initial data, which are well defined because A is functionally differentiable in the spacetime metric, and variations of the metric satisfying the linearized field equations are determined, up to diffeomorphisms, by the variations of the initial data. (See [Rei07] appendix C.) A may thus be replaced in (1) by a sum of smeared null initial data, yielding an equation entirely in terms of the variation δ' of the null initial data, and the Poisson brackets of these data.[29]

The requierment that $\delta_1' = \delta_1 - \pounds_{u_1}$ preserves the double null sheet character of $\mathcal{N}$ leaves considerable freedom in the choice of u_1. This freedom will be exploited to restrict the variations we have to consider still further. We will require

1 that the variations map the generators that lie in the boundary $\partial\mathcal{N}$ to

[29] As mentioned in a previous footnote, an alternative point of view is possible, in which the variations are not restricted to be null sheet preserving, but rather the definitions of the null data are extended to geometries in which $\mathcal{N}$ is not null. We will not adopt this point of view but let us sketch it here: Suppose δ is a, not necessarily null sheet preserving, variation. Recall that the action $\delta'\varphi(\theta, v)$ of a null sheet preserving component $\delta' = \delta - \pounds_u$ of δ on the a null datum $\varphi(\theta, v)$ on $\mathcal{N}$ is equal to action of δ on the same datum on a double null sheet $\mathcal{N}'$ that varies with the metric. A choice of this double null sheet suitable for δ_1 is $\mathcal{N}' = \mathcal{N}_{S_0 \cdot g}$, since it is defined for all variations in L_g. For δ_2 a suitable choice is $\mathcal{N}' = \tilde{\mathcal{N}}_{\partial\mathcal{N}}$, since it is defined for all variations in L_g^0 and corresponds to $u_2 = 0$ in a neighborhood of $\partial\mathcal{N}$. With this interpretation of the null data in the variations the explicit expression for the symplectic 2-form in terms of these data obtained in section 4.5 applies directly to any pair of variations $\delta_1 \in L_g$, $\delta_2 \in L_g^0$, whether they are null sheet preserving or not.

Condition (1) reduces to an equation on the Poisson bracket on the null data on $\tilde{\mathcal{N}}_{S_0}$ as follows: As we have seen, $\omega_{\mathcal{N}}[\{A,\cdot\}_\bullet, \delta]$ may be expressed in terms of the δ variations of data on $\tilde{\mathcal{N}}_{\partial\mathcal{N}}$ and the variations under $\{A,\cdot\}_\bullet$ of data on $\tilde{\mathcal{N}}_{S_0}$. Furthermore δA may be expressed as a sum of the δ variations of the data on $\tilde{\mathcal{N}}_{\partial\mathcal{N}}$, smeared with the functional derivatives of A by these data. Now note that the functional derivatives of A by the data on $\tilde{\mathcal{N}}_{\partial\mathcal{N}}$ and on $\tilde{\mathcal{N}}_{S_0}$ are in fact the same, because the variation of the metric produced by a variation of the data on $\tilde{\mathcal{N}}_{\partial\mathcal{N}}$ and that produced by the same variation of the data on $\tilde{\mathcal{N}}_{S_0}$ differ by a diffeomorphism, and A is diffeomorphism invariant. Thus $\{A,\cdot\}_\bullet$ can be expanded into a sum of the Poisson actions of the initial data on $\tilde{\mathcal{N}}_{S_0}$, smeared with the same functions (of θ and v) as appear in the expansion of δA in terms of variations of the data on $\tilde{\mathcal{N}}_{\partial\mathcal{N}}$.

themselves,

2 that they leave invariant the area density $\bar{\rho}$ in the fixed chart y_A on the truncation surface S_A of each branch,

and finally,

3 that they leave invariant a special chart constructed from the metric field in a spacetime neighborhood of each truncation surface S_A.

These charts, the a_L and a_R charts defined in subsection 4.3, will play an important role in the evaluation of the symplectic 2-form in terms of null data.

All these conditions already hold for δ'_2 because this variation leaves the entire metric field invariant in a spacetime neighborhood of $\partial\mathcal{N}$. They can be made to hold for δ'_1 by adding a suitable diffeomorphism generator, that is, by adjusting u_1: If δ'_1 perturbs the generators in $\partial\mathcal{N}$ then clearly a suitable diffeomorphism returns them to their unperturbed courses. If δ'_1 alters the y chart area density $\rho_y = \det[\partial\theta/\partial y]\rho$ at the endpoint of a generator on S_A then the generator can always be extended or shortened so that ρ_y at the new endpoint equals the unperturbed value of ρ_y at the old endpoint on S_A, because $\rho_y \propto v^2$ is nowhere stationary along the generator. The generators thus lengthened or shortened can then be mapped to the original generators of $\mathcal{N}_A$ by a diffeomorphism.

It remains only to ensure that the a_A chart is preserved by δ'_1 in a neighborhood of S_A. Clearly this can be done by adding a diffeomorphism generator to δ'_1. What has to be shown is that it can be done without violating the other conditions on δ'_1. Let us suppose then that δ'_1 preserves the double null sheet character of $\mathcal{N}$, and that it satisfies conditions 1 and 2.

As will be explained in subsection 4.3 the a_A chart is an extension to a spacetime region of a chart on $\mathcal{N}_A$, formed from the coordinates y^i_A, $r = v/\bar{v}$, and a fourth coordinate, u. The y^i label the generators, with each generator taking the values of y^i of its endpoint on S_A, while r labels the points within each generator. Finally, u is a coordinate transverse to the hypersurface swept out by the generators. It vanishes on the generators themselves.

On S_A itself the coordinates y^i_A are fixed by definition, and δ'_1 preserves them on $\partial\mathcal{N}_A - S_A - S_0$ because of condition 1. It also preserves $u = 0$ on $\mathcal{N}_A$ because it is null sheet preserving, implying that the generators remain in $\mathcal{N}_A$. Thus condition 3, that the variation leaves invariant all the a coordinates in a spacetime neighborhood of S_A, can be realized by adding a diffeomorphism generator which leaves $\partial\mathcal{N}_A$ invariant and maps $\mathcal{N}_A$ to itself (that is, one that corresponds to a vector field that vanishes on $\partial\mathcal{N}_A$ and is tangent to $\mathcal{N}_A$ on the remainder of $\mathcal{N}_A$). But such a diffeomorphism generator clearly preserves conditions 1 and 2, and the null sheet character of $\mathcal{N}_A$.

The null sheet preserving variations satisfying conditions 1,2, and 3 will be called *admissible* variations.[30] In subsection 4.4 the symplectic potetial $\Theta[\delta]$ is calculated in terms of our null data on admissible variations. Then, in subsection

[30] In [Rei08] a somewhat smaller set of variations was termed "admissible".

4.5, the symplectic 2-form is calculated from the symplectic potential via (29), again for admissible variations. This is possible because the commutator of admissible variations is also admissible.

4.2 The limited role of the diffeomorphism data

What is the role of the "diffeomorphism data" introduced in subsection 2.2? Recall that the free data defined in subsection 2.2 consists of the so called "v data", which are equivalent to Sachs' free null initial data, as well as the diffeomorphism data s_A and $\bar{v}_A$. These latter data constitute partial information about how the $v\theta$ charts, to which the v data are referred, are placed on $\mathcal{N}$.

The diffeomorphism data appear in the expression for the symplectic 2-form found in [Rei07] and used in [Rei08]. Indeed Poisson brackets are calculated for them. However, it was also argued in subsection 2.2 that the diffeomorphism data are not essential to the canonical formulation of general relativity in the domain of dependence of $\mathcal{N}$. They do not affect the spacetime geometry in the domain of dependence, nor the so called observables, which are functionals of the geometry, nor the Poisson brackets between these observables. Thus the condition (1), which ensures that the Poisson bracket on the data reproduces the Peierls brackets of the observables, ought not define brackets for the diffeomorphism data.

Here this expectation will be confirmed. It will be shown that the symplectic 2-form $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ does not depend on the variations of the diffeomorphism data if δ_2 vanishes in a neighborhood of $\partial\mathcal{N}$, and that (1) provides no information about the Poisson brackets of the diffeomorphism data.

The diffeomorphism data will nevertheless be retained in the present work. This is done mainly for consistency with [Rei08], which the present work underpins. In [Rei08] a strengthened version of (1), in which the test variation δ need not vanish in a neighborhood of $\partial\mathcal{N}$, is used to define a Poisson bracket on all the free data of subsection 2.2, including the diffeomorphism data. This strengthened condition requires an expression for $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ in terms of the null initial data valid for *all* admissible variations. It is this expression, which depends on the variations of the diffeomorphism data s_A, that is obtained in subsection 4.5.

Let us turn to the demonstration of the claims made above. Recall that when $\delta_2 \in L_g^0$ one may replace δ_1 and δ_2 in $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ by corresponding admissible variations without changing the value of $\omega_{\mathcal{N}}[\delta_1, \delta_2]$, and that the admissible variation corresponding to δ_2 still lies in L_g^0. Thus we may restrict our attention to admissible δ_1 and δ_2 without loss of generality.

The variations of the v data determine δg_{ab} and $\nabla_c \delta g_{ab}$ on $\mathcal{N}$ up to diffeomorphism generators. So they characterize δ_1 sufficiently for the calculation of the symplectic 2-form $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ when δ_1 and δ_2 are admissible variations in L_g and L_g^0 respectively. The situation is a little more subtle for δ_2. Since $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ is not invariant under the addition of non gauge diffeomorphism generators to δ_2, the variation under δ_2 of non gauge diffeomorphism degrees of freedom must be specified. The diffeomorphism data, s_A and $\bar{v}_A$, measure such degrees of freedom. However, because $\delta_2 g_{ab}$ is required to vanish in a spacetime neighbor-

hood of $\partial\mathcal{N}$ the variations of the v data under δ_2 in fact determine those of the diffeomorphism data modulo gauge. Thus ultimately $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ depends only on the variations under δ_1 and δ_2 of the v data, and of course the unperturbed values of the v data and of $\bar{v}_A$. (It does not depend on the unperturbed values of s_A because s_A can be set to any desired value by a diffeomorphism that maps $\mathcal{N}$ to itself, and $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ is invariant under such diffeomorphisms.)

How does this come about? Because δ_2 vanishes in a neighborhood of $\partial\mathcal{N}$ the y chart area density on S_A, $\bar{\rho}_A(y)$, is invariant under δ_2. Thus the variation of

$$\bar{v}_A(\theta) = \sqrt{\frac{\bar{\rho}_A(s_A(\theta)) \det[\partial s_A/\partial\theta]}{\rho_0}} \tag{52}$$

is determined by those of ρ_0 and s_A.[31]

It remains to show that the variations of s_A under $\delta_2 \in L_g^0$ are determined by those of the v data. In fact there is a trivial sense in which s_A can vary independently of the v data when there is enough symmetry. The field s_A depends on the choice of θ coordinates on S_0, which is a gauge choice in our formalism. If the spacetime geometry near $\mathcal{N}$ admits an isometry, a rotation, that maps $\mathcal{N}$ to itself, then it is possible to change s_A without changing the v data, by rotating the θ chart. But of course such a variation is pure gauge. It does not contribute to the symplectic 2-form because it does not change the spacetime metric components or their derivatives at any point of $\mathcal{N}$, and the symplectic 2-form depends only on the variations of the spacetime metric. Such gauge variations will be eliminated by holding the θ chart fixed in the variations we consider. As we will now see, once this restriction is imposed $\delta_2 s_A$ is indeed determined by the corresponding variations of the v data.

Suppose δ and $\tilde{\delta}$ are two admissible variations in L_g^0 that induce the same variations of v data. If the $v\theta$ charts on the braches of $\mathcal{N}$ are given then the v data determine the metric and its derivatives on each branch $\mathcal{N}_A$ up to diffeomorphisms that fix the points of $\mathcal{N}$. If the v data are given but the placement of the $v\theta$ charts is not specified, then there is of course an additional freedom in the metric corresponding to movements of this chart. It follows that $\delta g_{ab} - \tilde{\delta} g_{ab} = \pounds_\xi g_{ab}$ where the vector field ξ generates a diffeomorphism that maps $\mathcal{N}$ to itself, and furthermore that on $\mathcal{N}$ the field ξ reduces to the difference in the δ and $\tilde{\delta}$ variations of the v and θ coordinates: $\delta\theta^p - \tilde{\delta}\theta^p = \xi^p$, $\delta v - \tilde{\delta} v = \xi^v$.

This has two immediate consequences. First, since the θ chart (and of course also $v = 1$) is fixed on S_0, $\xi = 0$ there. Second, since $\delta - \tilde{\delta} \in L_g^0$, the diffeomorphism generated by ξ must reduce to an isometry in a neighborhood U of $\partial\mathcal{N}$. But since ξ vanishes on S_0, $S_0 \cap U$ is fixed under the isometry, as are its two future null normal directions. Because the area density $\rho = \rho_0 v^2$ is not constant along the generators, these future null directions cannot be rescaled isometrically. The isometry must preserve not only the directions but the vectors ∂_{v_A}.

[31] In fact, by definition all admissible variations leave $\bar{\rho}_A(y)$ invariant, so the variations of $\bar{v}_A$ can be eliminated from $\omega_{\mathcal{N}}[\delta_1, \delta_2]$ whenever δ_1 and δ_2 are admissible, even if neither lies in L_g^0. Precisely this will be done in our calculation of the symplectic 2-form on admissible variations.

In sum, the isometry preserves $S_0 \cap U$ and a complete basis of spacetime vectors at $S_0 \cap U$. The isometry is therefore trivial, that is $\xi = 0$, throughout U.[32] This implies in particular that $\delta\theta^p - \tilde{\delta}\theta^p = 0$ at S_A, so $\delta s_A = \tilde{\delta} s_A$.

Extending the preceeding argument one can conclude that the condition (1), $\delta A = \omega_{\mathcal{N}}[\{A, \cdot\}_\bullet, \delta]\ \forall \delta \in L_g^0$, does not define, nor impose any restriction on, the brackets of the diffeomorphism data, because brackets involving these data do not enter the condition: Because the observable A is diffeomorphism invariant by definition $\{A, \cdot\}_\bullet$ does not depend on $\{s, \cdot\}_\bullet$ or $\{\bar{v}, \cdot\}_\bullet$, and because δg_{ab} vanishes in a neighborhood of $\partial\mathcal{N}$, $\omega_{\mathcal{N}}[\{A, \cdot\}_\bullet, \delta]$ does not depend on $\{A, s\}_\bullet$ or $\{A, \bar{v}\}_\bullet$. The fact that the variation of the diffeomorphism data are determined by those of the v data under $\delta \in L_g^0$, and the gauge invariance of A, then imply that δs and $\delta\bar{v}$ do not enter (1) either.

How was it then possible to obtain the brackets of the diffeomorphism data in [Rei08]? In [Rei08] brackets were obtained for all the data, including the diffeomorphism data, by imposing a strengthened version of (1). The bracket was required to satisfy the conditions

$$\delta A = \omega_{\mathcal{N}}[\{A, \cdot\}_\bullet, \delta] \quad \forall \delta \in C \tag{53}$$

$$\{A, \cdot\}_\bullet \in C, \tag{54}$$

C being a subset of the admissible variations[33] containing the null sheet preserving variations in L_g^0 as a proper subset.[34] [35] These conditions define an essentially unique bracket on all the data. Of course any bracket satisfying the stronger condition (53) also satisfies the weaker condition (1), so the brackets of the v data given in [Rei08] are a solution to (1).

4.3 The a and b charts

Two types of special spacetime charts, called "a" charts and "b" charts, will be used. The charts b_L and b_R extend the $v\theta$ charts on $\mathcal{N}_L$ and $\mathcal{N}_R$ to charts on an open spacetime neighborhood of the interior, $S_0 - \partial S_0$, of S_0. Both are formed from the same coordinates $v^L, v^R, \theta^1, \theta^2$, but they differ in the ordering of these coordinates: $b_R^\mu = (v^L, v^R, \theta^1, \theta^2)$ and $b_L^\mu = (v^L, v^R, \theta^2, \theta^1)$. That is, the roles

[32] Isometries are rigid in any connected spacetime with a smooth non-degenerate metric: They are completely determined by their actions on one point of the spacetime and on the tangent space at that point. See [Wald84] p. 442 for a proof.

[33] In [Rei08] the term "admissible variation" is defined more narrowly than here and refers only to the variations in C.

[34] Recall that δ in (1) may be restricted to null sheet preserving variations in L_g^0 without weakening this condition, so the fact that C contains all these variations implies that (53) is at least as strong as (1).

[35] In [Rei08] one natural condition on the bracket is *relaxed*, namely the requierment that the changes in the metric on $\mathcal{N}$ be real. But the complex variations of the metric that are generated via the resulting bracket are special modes that represent shock waves that propagate along $\mathcal{N}$ and do not affect the metric on the interior of the domain of dependece of $\mathcal{N}$. A similar relaxation of the reality conditions on the data probably has to be made also to obtain a Poisson bracket satisfying (1). This ultimately seems to be a consequence of insisting on defining the Poisson bracket on all modes of the initial data, including these schock wave modes which are superflous for describing the interior of the domain of dependence.

of v^L and v^R are interchanged in the two charts, as are those of θ^1 and θ^2, so that the charts have the same orientation. The coordinates v^L, v^R, θ^1, and θ^2 are obtained from the $v\theta$ charts by setting $v^R = 1$ on $\mathcal{N}_L$ and $v^L = 1$ on $\mathcal{N}_R$, and then extending the functions v^L, v^R, θ^1, and θ^2 arbitrarily, but smoothly, off $\mathcal{N}$. Lowercase indices $\mu, \nu, ...$ from the latter part of the Greek alphabet will represent b coordinate indices.

The a_A chart, associated with the branch $\mathcal{N}_A$, is defined in much the same way as the b_A chart, but with the truncating 2-surface S_A playing the role of S_0. It consists of the ordered coordinates $a^\alpha_A = (u_A, r_A, y^1_A, y^2_A)$. y^1 and y^2 are constant on the generators of $\mathcal{N}_A$ and coincide on S_A with the fixed y chart already introduced to define the diffeomorphism datum s.[36] r is an area parameter along the generators like v, but normalized to 1 on S_A, so $r = \sqrt{\rho_y/\bar{\rho}} = v/\bar{v}$, where ρ_y is the area density on cross sections of $\mathcal{N}_A$ in the y chart, and $\bar{\rho}$ is the area density on S_A in this chart. r, y^1, and y^2 are extended off $\mathcal{N}_A$ by holding them constant on the null geodesics normal to the equal r cross sections of $\mathcal{N}_A$ and transverse to $\mathcal{N}_A$. Finally u is a parameter along these geodesics set to 0 on $\mathcal{N}_A$ and chosen such that $\partial_u \cdot \partial_r = -1$. Greek lowercase indices $\alpha, \beta, ...$ from the beginning of the alphabet will represent a coordinate indices.

On $\mathcal{N}_A$ the transformation between the a and b charts is quite simple:

$$r = v/\bar{v}(\theta) \quad y^i = s^i(\theta) \quad u = 0. \tag{55}$$

In the a chart the spacetime line element at $\mathcal{N}_A$ takes the form[37]

$$ds^2 = -2dudr + h_{ij}dy^i dy^j = -2dudr + r^2\bar{\rho}e_{ij}dy^i dy^j. \tag{56}$$

The spacetime line element at S_0 is also simple in the b chart. It is

$$ds^2 = 2\chi dv^L dv^R + h_{pq}d\theta^p d\theta^q, \tag{57}$$

with $\chi = \partial_{v^L} \cdot \partial_{v^R}$.

It will be necessary to have control over the orientations of the charts we have defined. The sign of the integral of a form over a manifold depends on the orientation of the manifold.[38] Given this orientation the integral of the form can be reduced to an iterated definite integral by choosing a chart x oriented coherently with the manifold, and expressing the integrand as a multiple of the coordinate volume form: $f\, dx^1 \wedge dx^2 \wedge ... \wedge dx^n$, where f is a suitable function.

[36] To lighten notation the branch index, A, will usually be supressed when there is little risk of confusion.

[37] The e_{ij} are the y chart components of the conformal 2-metric, which is a (2-dimensional) weight -1 tensor density.

[38] Two overlapping charts are said to be coherently oriented if the transformation between them has positive Jacobian determinant. The orientation of a manifold is defined by the choice of a coherently oriented atlas on the manifold. (A manifold may or may not admit a coherently oriented atlas. If it does it is orientable, and if it does not it is non-orientable.) See [CDD82]. The orientation of a chart on an oriented manifold is said to be positive, or to match that of the manifold, if the chart is coherently oriented with the atlas that defines the orientation of the manifold.

The integral is then $\int f dx^1 dx^2 ... dx^n$, with the integration over each x^s running from lesser to greater values of x^s. (See [CDD82].)

An orientation will be chosen, once and for all, for spacetime (or at least a neighborhood of $\mathcal{N}$). Which orientation is chosen does not matter, because whichever choice is made, the sign of the 4-volume form ε will be chosen so that its integral over a spacetime region is positive. The value of the action (21) is thus independent of the orientation of spacetime chosen.

The a charts will be positively oriented, that is, their orientations will be chosen to match that of spacetime. This can be achieved by choosing a suitable orientation for the y_A chart on S_A.

We shall take $\mathcal{N}$ to be future oriented. It is with this convention that equation (1) ensures that the Poisson bracket on initial data reproduces the Peierls bracket [Rei07]. A chart x^1, x^2, x^3 on a non-timelike hypersurface is future oriented if a spacetime chart, t, x^1, x^2, x^3, formed from the x coordinates and a time coordinate t which is constant on the hypersurface and increasing toward the future, is positively oriented.

If r increases toward the future on $\mathcal{N}_A$ then u must also, because $g_{ur} = -1$ and the metric is assumed to have signature $-+++$. u may therefore be taken as the time coordinate in the preceeding definition, and u, r, y^1, y^2 as the combined positively oriented spacetime chart. Thus, in this case, r, y^1, y^2 is a future oriented chart on $\mathcal{N}_A$, and therefore matches the orientation of this manifold. By a similar argument, if r decreases toward the future, then $-u, -r, y^1, y^2$ is a positively oriented chart with $-u$ a time coordinate increasing toward the future, so $-r, y^1, y^2$ is a future oriented chart on $\mathcal{N}_A$. In either case the coordinate along the generators of the future oriented chart increases from S_0 to S_A. Thus

$$\int_{\mathcal{N}_A} f dr \wedge dy^1 \wedge dy^2 = \int_{S_A} d^2y \int_{r_0}^{1} dr f, \tag{58}$$

where the y integrals run from lesser to greater values of these coordinates, or, equivalently, d^2y is interpreted as the positive euclidean coordinate measure on S_A defined by the y chart. On each generator $r_0 = 1/\bar{v}$ is the value of r at S_0.

A future orientation can be defined on the two dimensional cross sections of $\mathcal{N}_A$ in an entirely analogous manner, with $\mathcal{N}_A$ now playing the role of spacetime in the preceeding definition. The y chart gives precisely this future orientation to S_A if $\mathcal{N}_A$ is future oriented in spacetime. This is also the orientation that S_A has as part of the boundary of $\mathcal{N}_A$.

The θ_A chart will be oriented coherently with the y_A chart. Therefore, if $S_0^{(A)}$ is S_0 oriented coherently with $\partial\mathcal{N}_A$, and thus past oriented with respect to $\mathcal{N}_A$, then

$$\int_{S_0^{(A)}} d\theta_A^1 \wedge d\theta_A^2 = -\int_{S_0} d^2\theta. \tag{59}$$

Since $S_0^{(L)}$ and $S_0^{(R)}$ have opposite orientations it follows at once that the charts θ_R and θ_L must be oppositely oriented. If (θ^1, θ^2) is coherently oriented with y_R then $\theta_R^p = (\theta^1, \theta^2)$ and $\theta_L^p = (\theta^2, \theta^1)$ satisfy our requirements. These correspond

to the b charts $b^\mu_R = (v^L, v^R, \theta^1, \theta^2)$ and $b^\mu_L = (v^R, v^L, \theta^2, \theta^1)$ defined earlier. The use of these two b charts, instead of just one, b_R say, makes possible a completely symmetrical treatment of the two branches.

4.4 The symplectic potential in terms of the free null data

According to (28) the contribution to the symplectic potential of a branch $\mathcal{N}_A$ of $\mathcal{N}$ is

$$\Theta_A[\delta] = -\frac{1}{8\pi G}\int_{\mathcal{N}_A} \delta\Gamma^{[c}_{cb} g^{a]b}\, \varepsilon_{a\cdots}, \tag{60}$$

with the whole symplectic potential given by $\Theta_\mathcal{N} = \Theta_L + \Theta_R$. Our task is to rewrite $\Theta_\mathcal{N}[\delta]$ in terms of our free initial data for admissible variations δ. Taking the curl of this potential then yields the symplectic 2-form in terms of these data and variations.

In the following only Θ_R will be computed explicitly. Θ_L is entirely analogous, except that τ is replaced by $-\tau$ because exchanging L and R in the definition (11) of τ produces an expresion equal to $-\tau$.

It will be convenient to decompose the variation δ into the sum of a diffeomorphism generator $\pounds_\xi$ that accounts for the displacement of the a_R chart under δ, and a variation $\delta^a = \delta - \pounds_\xi$, that leaves this a chart fixed. As is explained in detail in appendix A, $\delta^a g$ is the part of the variation of the metric arising from changes of the metric components in the a chart: In this chart $[\delta^a g]_{\alpha\beta}(a) = \delta[g_{\alpha\beta}(a)]$. The remainder, $\pounds_\xi g$, is of course the part of the variation arising from the shift of the a chart. The corresponding decomposition of Θ_R,

$$\Theta_R[\delta] = \Theta_R[\delta^a] + \Theta_R[\pounds_\xi]. \tag{61}$$

neatly separates the contribution from the variations of the bulk datum, the conformal 2-metric e, and the variations of the surface data on S_0.

$\Theta_R[\delta^a]$ depends only on the variation of e. Indeed, in the a chart the metric at $\mathcal{N}_R$ is restricted to the form

$$g_{\alpha\beta} = \begin{bmatrix} 0 & -1 & \\ -1 & 0 & \\ & & h_{ij} \end{bmatrix}, \quad g^{\alpha\beta} = \begin{bmatrix} 0 & -1 & \\ -1 & 0 & \\ & & h^{ij} \end{bmatrix}, \tag{62}$$

with h^{ij} the inverse of h_{ij}. (See (56).) Furthermore $h_{ij} = r^2\bar{\rho}e_{ij}$. Since $\bar{\rho}$, the y chart area density on S_R, is invariant under admissible variations, e_{ij} is the only degree of freedom that can vary in $g_{\alpha\beta}$.

$\Theta_R[\pounds_\xi]$, on the other hand, is a surface integral: By (44)

$$\Theta_R[\pounds_\xi] = -\frac{1}{16\pi G}\int_{\partial\mathcal{N}_R} \nabla^a\xi^b\, \varepsilon_{ab\cdots} \tag{63}$$

In fact the integral reduces to one over S_0, because the integrand vanishes elsewhere: Since δ^a preserves the a chart ξ is determined by

$$\delta a^\alpha = \delta^a a^\alpha + \pounds_\xi a^\alpha = \xi^\alpha. \tag{64}$$

On S_R both the coordinates a^α and their gradients are invariant under admissible variations, so ξ and $\nabla\xi$ vanish there. There is thus no contribution to (63) from S_R. The fact that the contribution from $\partial\mathcal{N}_R - S_R - S_0$ also vanishes is most easily understood by expressing the integrand of (63) on this surface in terms of a chart components. The pullback of du to $\partial\mathcal{N}_R - S_R - S_0$ vanishes, and those of dy^1 and dy^2 are linearly dependent, so the pullback of $\nabla^a\xi^b\,\varepsilon_{ab\cdot\cdot}$ is equal to the pullback of $2\nabla^{[u}\xi^{i]}\,\varepsilon_{uirj}dr\wedge dy^j$. But

$$\nabla^{[u}\xi^{i]} = g^{u\alpha}g^{i\beta}\nabla_{[\alpha}\xi_{\beta]} = -h^{ik}\partial_{[r}\xi_{k]}. \tag{65}$$

Since admissible variations are null sheet preserving, $\delta u = 0$ on $\mathcal{N}_R$. Thus $\xi_r = -\xi^u = 0$, and it follows that $\partial_k\xi_r = 0$. Since admissible variations preserve the generators on $\partial\mathcal{N}_R - S_R - S_0$, $\delta y^i = 0$ there. Thus $\xi_k = h_{ki}\xi^i = 0$, and $\partial_r\xi_k = 0$, which establishes the claim. The diffeomorphism term is therefore

$$\Theta_R[\pounds_\xi] = -\frac{1}{16\pi G}\int_{S_0^{(R)}} \nabla^a\xi^b\,\varepsilon_{ab\cdot\cdot}, \tag{66}$$

where $S_0^{(R)}$ is S_0 oriented coherently with $\partial\mathcal{N}_R$.

The form of the vector field ξ can be restricted quite a bit by gauge fixing the variations further. In particular one can ensure that the variations leave the b chart fixed in a spacetime neighborhood of $S_0 - \partial S_0$. First one adds a diffeomorphism generator $\pounds_w$ to each variation so that the generators, and the v parameter on these, are invariant under the total variation within some neighborhood of S_0. This can be achieved with a diffeomorphism $\pounds_w$ which is pure gauge, that is, without affecting the value of the symplectic 2-form: Recall that admissible variations already preserve the double null sheet character of the fixed manifold $\mathcal{N}$, and map the generators in $\partial\mathcal{N}$ to themselves. Thus w is tangent to $\mathcal{N}$ and to the generators on $\partial\mathcal{N}$ within a neighborhood of S_0. We will set w to zero in neighborhoods of S_L and S_R and tangent to the generators on $\partial\mathcal{N}$ wherever it is non-zero. Then the change in the symplectic potential due to the addition of $\pounds_w$, $\Theta_{\mathcal{N}}[\pounds_w]$, vanishes, on S_L and S_R because w vanishes in a neighborhood of these surfaces, and on $\partial\mathcal{N} - S_L - S_R$ by the argument of the preceeding paragraph. Alternatively, one may note that the addition of $\pounds_w$ to δ does not affect δ^a but does transform $\xi \to \xi + w$. Since the new variation $\delta + \pounds_w$ is still admissible, the contribution to $\Theta_R[\pounds_\xi]$ from $\partial\mathcal{N}_R - S_R - S_0$ remains zero. The contribution from S_0, (66), is affected by the addition of of w to ξ, but the sum,

$$\Theta_R[\pounds_{\xi_R}] + \Theta_L[\pounds_{\xi_L}] = \frac{1}{16\pi G}\int_{S_0^{(R)}} \nabla^a[\xi_L - \xi_R]^b\,\varepsilon_{ab\cdot\cdot}, \tag{67}$$

is not because w cancels out in the difference $\xi_L - \xi_R$. (The minus sign is due to the fact that the orientation of $S_0^{(L)}$ is opposite to that of $S_0^{(R)}$.)

To ensure that the b chart is fixed under the gauge fixed variations the θ^1, θ^2 must be set equal to fixed coordinates on S_0. (Recall that the choice of the θ chart is a gauge degree of freedom in our formalism, additional to the spacetime diffeomorphism gauge freedom.) This fixes the b coordinates on $\mathcal{N}$

in a neighborhood of S_0. They may then be smoothly extended to a fixed chart on a spacetime neighborhood of the interior of S_0.

Once this gauge fixing has been carried out ξ simply measures the variation of the transformation from the b chart to the a chart (within the domain in which the b chart is fixed). On $\mathcal{N}_R$ the transformation between these charts is (55)

$$r = v/\bar{v}(\theta), \quad y^i = s^i(\theta), \quad u = 0. \tag{68}$$

Thus

$$\xi = \delta r \partial_r + \delta y^i \partial_{y^i} = -r\delta \ln \bar{v}\, \partial_r + \delta s^i \partial_{y^i}, \tag{69}$$

which is completely determined by the variations of the S_0 data s^i and $\bar{v}$. Recall that ξ is tangent to $\mathcal{N}_R$, since admissible variations preserve $u = 0$ on $\mathcal{N}_R$.

Let us evaluate the bulk term,

$$\Theta_R[\delta^a] = -\frac{1}{8\pi G}\int_{\mathcal{N}_R} \delta^a \Gamma^{[\alpha}_{\alpha\gamma} g^{\beta]\gamma} \varepsilon_{\beta\cdots} \tag{70}$$

, in the symplectic potential in terms of the initial data. It is convenient to use the a chart, since the a coordinates are fixed under δ^a, and the a components of the variation under δ^a of a field is simply the variation under δ of the a components of the field. Thus for example

$$[\delta^a \Gamma]^\alpha_{\beta\gamma}(a) = \delta[\Gamma^\alpha_{\beta\gamma}(a)], \tag{71}$$

where the variation δ on the right hand side is of the connection coefficients $\Gamma^\alpha_{\beta\gamma}$ evaluated at a fixed a coordinate point a.

The form (62) of the metric in the a chart implies that $\sqrt{-g} = \sqrt{det[h_{ij}]} \equiv \rho_y$ and therefore, since the a chart is positively oriented, that $\varepsilon = \rho_y\, du \wedge dr \wedge dy^1 \wedge dy^2$. Pulling the last three indices of this 4-volume form back to $\mathcal{N}_R$ one obtains

$$\varepsilon_{\beta\cdots} = \rho_y\, \delta^u_\beta\, dr \wedge dy^1 \wedge dy^2. \tag{72}$$

The integrand in (70) thus reduces to $dr \wedge dy^1 \wedge dy^2$ times the function

$$\begin{aligned}
\delta^a \Gamma^{[\alpha}_{\alpha\gamma} g^{u]\gamma} \rho_y &= \frac{1}{2}\{-\delta^a \Gamma^\alpha_{\alpha r} + g^{\alpha\gamma}\delta^a \Gamma_{r\alpha\gamma}\}\rho_y && (73)\\
&= -\frac{1}{2}\{\delta^a \partial_r \ln\sqrt{-g} + 2\delta^a \Gamma_{rur} - h^{ij}\delta^a \Gamma_{rij}\}\rho_y && (74)\\
&= -\frac{1}{2}\{\delta^a \partial_r \ln \rho_y + \delta^a \partial_u g_{rr} + \frac{1}{2} h^{ij}\delta^a \partial_r h_{ij}\}\rho_y. && (75)
\end{aligned}$$

Now recall that $\rho_y(y, r) = r^2 \bar{\rho}(y)$ and that $\delta^a \bar{\rho} = 0$ for admissible variations. It follows that

$$\delta^a \partial_r \ln \rho_y = \delta^a 2/r = 0. \tag{76}$$

and

$$\frac{1}{2}\, h^{ij}\delta^a[\partial_r h_{ij}] = \delta^a \partial_r \ln \rho_y - \frac{1}{2}\partial_r[\rho_y\, e_{ij}]\delta^a \frac{e^{ij}}{\rho_y} = -\frac{1}{2}[\partial_r e_{ij}]\delta^a e^{ij}. \tag{77}$$

(Here e^{ij} is the inverse of e_{ij},[39] and the fact that, for any variation Δ, $e_{ij}\Delta e^{ij} = -\Delta \ln \det[e_{ij}] = -\Delta 0 = 0$ has been used.)

The remaining, middle, term in (75) is proportional to $\delta^a \Gamma^r_{rr}$, since on $\mathcal{N}_R$

$$\partial_u g_{rr} = 2\Gamma^r_{rr}. \tag{78}$$

Because $\mathcal{N}_R$ is oriented toward the future the form integral (70) reduces to an iterated definite integral according to the formula (58). Substituting in our results on the integrand one obtains

$$\Theta_R[\delta^a] = -\frac{1}{16\pi G}\int_{S_R} d^2y\,\bar{\rho}\int_{r_0}^{1}\frac{1}{2}r^2\partial_r e_{ij}\delta^a e^{ij} - 2r^2\delta^a\Gamma^r_{rr}dr, \tag{79}$$

where d^2y is the positive euclidean coordinate measure associated with the y chart. The first term in the integrand is expressed in terms of the bulk datum e. The second term combines with a term in the surface contribution to the symplectic potential to form a total variation, which may be dropped from the potential without affecting the symplectic 2-form.

Let us now turn to the surface term, $\Theta_R[\pounds_\xi]$, in the symplectic potential. Here it is convenient to work with the b chart, since the S_0 data is defined in terms of this chart.

Our first task will be to calculate the 4-volume form in the b chart. Because $n_L = \partial_{v^L}$ and $n_R = \partial_{v^R}$ are null and normal to S_0 the spacetime line element at S_0 takes the form (57)

$$ds^2 = 2\chi dv^L dv^R + h_{pq}d\theta^p d\theta^q, \tag{80}$$

with $\chi = n_L \cdot n_R$. But the sign of χ depends on the direction in which v^L and v^R increase. Recall the definition $\sigma_A = 1$ if v^A increases to the future, and -1 if it decreases. Then $\sigma_A n_A$ is future directed and $\sigma_L\sigma_R\chi$ is negative. Indeed $\chi = -\sigma_L\sigma_R\, e^{-\lambda}$.

The chart $(\sigma_L v^L, \sigma_R v^R, \theta^1, \theta^2)$ has the same, positive, orientation as the a_R chart, so the 4-volume form is

$$\begin{aligned}
\varepsilon &= |\chi|\,\rho_0\,\sigma_R\sigma_L\, dv^L\wedge dv^R\wedge d\theta^1\wedge d\theta^2 &(81)\\
&= -\chi\,\rho_0\, dv^L\wedge dv^R\wedge d\theta^1\wedge d\theta^2 &(82)\\
&= -\frac{\rho_0}{\chi}n_R\wedge n_L\wedge d\theta^1\wedge d\theta^2. &(83)
\end{aligned}$$

In the last line the n_A denote the 1-forms $n_{L\,a} = \chi\partial_a v^R$ and $n_{R\,a} = \chi\partial_a v^L$, obtained by lowering the indices of the tangent vectors n^a_A with the metric.

In (66) the first two indices of ε_{abcd} are contracted and the last two indices are pulled back to S_0. When thus pulled back $\varepsilon_{ab\cdot\cdot}$ becomes

$$\varepsilon_{ab\cdot\cdot} = -2\frac{\rho_0}{\chi}\,n_{R\,[a}n_{L\,b]}\,d\theta^1\wedge d\theta^2. \tag{84}$$

[39] The e^{ij} are the y coordinate components of the inverse conformal 2-metric defined earlier, which is a weight 1, 2-dimensional tensor density.

Substituting this expression into (66) yields

$$\Theta_R[\pounds_\xi] = \frac{1}{16\pi G}\int_{S_0} \frac{1}{\chi}\{n_R \cdot \nabla_{n_L}\xi - n_L \cdot \nabla_{n_R}\xi\}\,\rho_0\, d^2\theta, \tag{85}$$

with $d^2\theta$ the positive euclidean measure defined by the θ chart on S_0. The formula (59), which takes into account the orientation of S_0 employed in (66), has been used to turn (66) into a definite integral.

The derivative along n_L may be eliminated in favour of a variation of χ:

$$\delta\chi = n_L^a n_R^b \delta g_{ab} \;=\; n_L^a n_R^b[\delta^a g_{ab} + \pounds_\xi\, g_{ab}] \tag{86}$$

$$\;=\; n_L^\alpha n_R^\beta\, \delta^a g_{\alpha\beta} + 2 n_L^a n_R^b \nabla_{(a}\xi_{b)}. \tag{87}$$

But $n_R^\beta = \partial_{v^R} a^\beta = 1/\bar{v}\,\delta_r^\beta$, and (by (62)) $g_{\alpha r} = -\delta_\alpha^u$, which is of course invariant, so $n_L^\alpha n_R^\beta\, \delta^a g_{\alpha\beta} = 0$ and

$$\delta\chi = n_R \cdot \nabla_{n_L}\xi + n_L \cdot \nabla_{n_R}\xi. \tag{88}$$

The integrand of (85) is thus equal to

$$\frac{1}{\chi}[\delta\chi - 2n_L \cdot \nabla_{n_R}\xi]\rho_0 = -\rho_0\delta\lambda - 2\frac{\rho_0}{\chi} n_L \cdot \nabla_{n_R}\xi, \tag{89}$$

since $\lambda = -\ln|\chi|$.

Notice that the θ components of ξ, $\xi^p = \xi \lrcorner\, d\theta^p = \delta s^i\, \partial_{y^i}\theta^p$, are independent of v. This means that $\xi_\perp \equiv \xi^p\partial_p$ is Lie dragged along $n_R = \partial_v$:

$$0 = \pounds_{n_R}\xi_\perp = \nabla_{n_R}\xi_\perp - \nabla_{\xi_\perp} n_R, \tag{90}$$

for in the b chart the Lie derivative along n_R reduces to simply the v partial derivative of the b components of $\xi_\perp$.

The second term in (89) may therefore be expanded according to

$$2n_L \cdot \nabla_{n_R}\xi = 2n_L \cdot \nabla_{n_R}[\xi_\perp + \xi^v n_R] = 2n_L \cdot \nabla_{\xi_\perp} n_R + 2\xi^v n_L \cdot \nabla_{n_R} n_R + 2\chi d_{n_R}\xi^v \tag{91}$$

The first term in this expansion is linear in the twist τ. By the definition (11) of τ

$$2n_L \cdot \nabla_{\xi_\perp} n_R = d_{\xi_\perp}\chi + \chi\xi_\perp \lrcorner\, \tau = -\chi\xi_\perp \lrcorner\, [d\lambda - \tau]. \tag{92}$$

The second and third terms in (91) are proportional to the values on S_0 of ξ^v and $d_{n_R}\xi^v = \partial_v\xi^v$ respectively. In a neighborhood of S_0 $\delta v = 0$, so $\delta^a v = -\pounds_\xi v = -\xi^v$ there. It follows that

$$\xi^v = -\delta^a v = -\delta^a[r/r_0] = v\,\delta^a \ln r_0, \tag{93}$$

and therefore on S_0

$$\xi^v = \partial_v \xi^v = \delta^a \ln r_0. \tag{94}$$

The second term in (91) also contains a factor $n_L \cdot \nabla_{n_R} n_R$. Since n_R is tangent to a geodesic and $n_R = \partial_v = r_0 \partial_r$

$$n_L \cdot \nabla_{n_R} n_R = n_L \cdot r_0^2 \nabla_{\partial_r} \partial_r = n_L \cdot r_0 \Gamma^r_{rr} n_R = \chi r_0 \Gamma^r_{rr}. \tag{95}$$

The second and third terms are the ones that will combine with terms from the bulk contribution to the symplectic potential to form a total variation.

Substituting our results into (85) and adding the bulk contribution $\Theta_R[\delta^a]$ we obtain the complete symplectic potential of $\mathcal{N}_R$:

$$\begin{aligned}\Theta_R[\delta] \quad = \quad & -\frac{1}{16\pi G}\left\{ \int_{S_0} [\delta\lambda - \xi_\perp \lrcorner\, (d\lambda - \tau) + 2\delta^a \ln r_0 (1 + r_0 \Gamma^r_{rr})] \rho_0 \, d^2\theta \right. \\ & \left. + \int_{S_R} d^2 y \, \bar{\rho} \int_{r_0}^1 [\frac{1}{2} r^2 \partial_r e_{ij} \delta^a e^{ij} - 2r^2 \delta^a \Gamma^r_{rr}] dr \right\} \end{aligned} \tag{96}$$

The dependence on Γ^r_{rr} can be eliminated. As has already been pointed out, adding the variation of a functional of the data to the symplectic potential does not affect its curl, the symplectic 2-form. Thus we are free to subtract from $\Theta_R[\delta]$ the variation

$$\begin{aligned} & \delta\left[\frac{1}{8\pi G} \int_{S_R} d^2 y \, \bar{\rho} \int_{r_0}^1 [r + r^2 \Gamma^r_{rr}] dr\right] \\ & = \quad \frac{1}{8\pi G}\left\{ \int_{S_R} d^2 y \, \bar{\rho} \int_{r_0}^1 r^2 \delta^a \Gamma^r_{rr} dr - \int_{S_0} d^2\theta \, \rho_0 \, \delta^a \ln r_0 \, (1 + r_0 \Gamma^r_{rr}) \right\}. \end{aligned} \tag{97}$$

Recall that the variation $\delta[F]_a$ of the a chart components of a field F, at a fixed a coordinate point, is given by the a chart components of $\delta^a F$. This is the reason for the appearance of δ^a in (97). (See (129) of appendix A.) Use has also been made of the fact that $r_0^2 \bar{\rho}$ is the y chart area density on S_0, so $r_0^2 \bar{\rho} \, d^2 y = \rho_0 \, d^2\theta$.

We will therefore take as the symplectic potential of $\mathcal{N}_R$

$$\Theta'_R[\delta] = -\frac{1}{16\pi G}\left\{ \int_{S_0} [\delta\lambda - \xi_\perp \lrcorner\, (d\lambda - \tau)] \rho_0 \, d^2\theta + \frac{1}{2} \int_{S_R} d^2 y \, \bar{\rho} \int_{r_0}^1 r^2 \partial_r e_{ij} \delta^a e^{ij} dr \right\}. \tag{98}$$

Proceeding in exactly the same way an analogous expression is obtained for Θ'_L, the symplectic potential of $\mathcal{N}_L$, with the one difference that τ is replaced with $-\tau$ since interchanging L and R maps τ to $-\tau$.

Equation (98) and its L branch analog provide an expression for the symplectic potential entirely in terms of our free null initial data. It depends on the v data ρ_0, λ, and τ on S_0, and on $\xi_{\perp\, A} = \delta s^i_A \, \partial_{y^i_A}$ there. It further depends on the a_A chart conformal 2-metric e_{ij} on $\mathcal{N}_A$, on the (invariant) y_A chart area density $\bar{\rho}_A = |det \frac{\partial s^i_A}{\partial \theta^p}|^{-1} \rho_0 \bar{v}_A^2$ on S_A, and on $r_{A\,0} = 1/\bar{v}_A$. Note that the transformation from the b chart conformal metric $e_{pq}(v,\theta)$, which is one of our data, to $e_{ij}(r,y)$ is determined by the transformation (68) from the b chart to the a_A

chart on $\mathcal{N}_A$, which in turn is determined by the diffeomorphism data s_A and $\bar{v}_A$.

We have achieved our goal of expressing the symplectic potential in terms of the null initial data of subsection 2.2. It turns out however that a symplectic potential that is in some ways more useful is obtained by replacing the datum τ by two new data

$$\tilde{\tau}_{R\,i} \equiv \rho_0(d\lambda - \tau) \lrcorner\, \partial_{y^i_R} \quad \text{and} \quad \tilde{\tau}_{L\,j} \equiv \rho_0(d\lambda + \tau) \lrcorner\, \partial_{y^j_L}. \tag{99}$$

These are the coefficients of δs^i_R and δs^j_L respectively in the surface term of the symplectic potential. In terms of these new data

$$\Theta'_R[\delta] = -\frac{1}{16\pi G}\left\{\int_{S_0} \rho_0\, \delta\lambda - \tilde{\tau}_{R\,i}\delta s^i_R\, d^2\theta + \frac{1}{2}\int_{S_R} d^2y\, \bar{\rho} \int_{r_0}^1 r^2 \partial_r e_{ij} \delta^a e^{ij} dr\right\}, \tag{100}$$

and Θ'_L is given by a completely analogous expression. (In particular $\tilde{\tau}_L$ enters Θ'_L in precisely the same way as $\tilde{\tau}_R$ enters Θ'_R, since the difference in the sign with which τ enters Θ'_L and Θ'_R has been absorbed into the definitions of $\tilde{\tau}_R$ and $\tilde{\tau}_L$.) In principle $\tilde{\tau}_R$ and $\tilde{\tau}_L$ are related by the equation

$$\tilde{\tau}_{R\,i} ds^i_R + \tilde{\tau}_{L\,j} ds^j_L = 2\rho_0 d\lambda. \tag{101}$$

However, we shall extend the phase space by taking $\tilde{\tau}_R$ and $\tilde{\tau}_L$ to be independent, and then treat (101) as a constraint which defines our original phase space. This constraint generates the gauge transformations of the θ chart [Rei07]. Thus, in the extended phase space, without the constraint, these transformations are not gauge, and the symplectic 2-form is in fact non-degenerate.

The introduction of constrained variables seems a step backward with respect to our aim of a canonical description in terms of free data, but the constraint introduced brings no real complications. Indeed, (101) may be solved easily for $\tilde{\tau}_R$. If the θ chart is then fixed via the gauge condition $s_R = \mathrm{id}$ (i.e. $\theta = y_R$), then the physical phase space is parametrized by the remaining data, and the Dirac brackets of these remaining data are equal to their brackets in the extended phase space [Rei07].

A non degenerate symplectic form can only be achieved by either extending the phase space as we do, or by gauge fixing the θ chart. The use of an unfixed, arbitrary, θ chart, has made it possible to treat the two branches of $\mathcal{N}$ autonomously and symetrically. This is also possible for some gauge fixed θ charts. For instance one could take the θ^p to be isothermal coordinates of the metric on S_0. This gauge choice has the drawback that isothermal coordinates depend non-locally on the metric. As a result the Dirac bracket, unlike the extended phase space Poisson bracket, does not always vanish between data on distinct generators. Other gauge fixings which avoid this complication can be defined, but all the same, leaving θ unfixed and working with the extended phase space seems the simplest choice.

4.5 The symplectic 2-form in terms of the free null data

The contribution of the hypersurface $\mathcal{N}_R$ to the symplectic form is

$$\omega_R[\delta_1, \delta_2] = \delta_1 \Theta'_R[\delta_2] - \delta_2 \Theta'_R[\delta_1] - \Theta'_R[[\delta_1, \delta_2]]. \tag{102}$$

Here this expression will be evaluated in terms of the free null initial data for admissible variations δ_1 abd δ_2. Since $[\delta_1, \delta_2]$ is also admissible the symplectic potential is needed only on admissible variations. Equation (100) for Θ'_R therefore provides a sufficient basis for the calculation.

The first term of Θ'_R in (100) is a surface term, an integral over S_0, while the second term is a bulk term, an integral over $\mathcal{N}_R$. As a result ω_R also consists of bulk and surface terms. The bulk term in ω_R is obtained by varying the bulk term in Θ'_R with r_0 held fixed. It is

$$\frac{1}{32\pi G} \int_{S_R} d^2y \, \bar{\rho} \int_{r_0}^{1} r^2 \delta_1^a e^{ij} \partial_r \delta_2^a e_{ij} \, dr - (1 \leftrightarrow 2). \tag{103}$$

Since r_0 is held fixed the domain of integration does not vary in the a chart. The variation of the integral is therefore just the integral of the variation of the integrand in this chart, that is, of δ^a of the integrand. (See (129) of appendix A.)

In terms of the $v\theta$ chart the bulk term in ω_R may be written as

$$\frac{1}{32\pi G} \int_{S_0} d^2\theta \, \rho_0 \int_1^{\bar{v}} v^2 \delta_1^a e^{pq} \partial_v \delta_2^a e_{pq} \, dv - (1 \leftrightarrow 2). \tag{104}$$

(Note that the transformation between y and θ components is independent of r, so it may be freely moved through the derivative, ∂_r, in (103).)

The surface contribution to ω_R comes both from the surface term in Θ'_R and from the variation of r_0 in the bulk term of Θ'_R. The surface term in Θ'_R yields

$$\frac{1}{16\pi G} \int_{S_0} \delta_1 \lambda \, \delta_2 \rho_0 + \delta_1 \tilde{\tau}_{R\,i} \, \delta_2 s_R^i \, d^2\theta - (1 \leftrightarrow 2). \tag{105}$$

The variation of r_0 in the bulk term in Θ'_R produces

$$\begin{aligned} &\frac{1}{32\pi G} \int_{S_R} \delta_1 [r_0]_y [r^2 \partial_r e_{ij} \delta_2^a e^{ij}]_{r=r_0} \bar{\rho} \, d^2y - (1 \leftrightarrow 2) \\ &= \frac{1}{32\pi G} \int_{S_0} \delta_1^a \ln r_0 \, \partial_v e_{pq} \delta_2^a e^{pq} \rho_0 \, d^2\theta - (1 \leftrightarrow 2), \end{aligned} \tag{106}$$

where $\delta[r_0]_y = \delta^a r_0$ is the variation of the scalar $r_0(y)$ at constant y. (See (129).)

In (106) δ^a may be replaced by the variation $\delta^y \equiv \delta - \pounds_{\xi_\perp} = \delta^a + \pounds_{\xi^v \partial_v}$, associated with the hybrid chart (v^L, v^R, y^1, y^2). Clearly $\delta^a \ln r_0 = \delta^y \ln r_0$ since r_0 only depends on y. Furthermore, as will soon be demonstrated

$$\pounds_{\xi^v \partial_v} e^{pq} = \xi^v \partial_v e^{pq}. \tag{107}$$

Substituting these two relations into the integrand of (106), and taking into account that by (94) $\xi^v = \delta^a \ln r_0 = \delta^y \ln r_0$ on S_0, one obtains

$$\delta_1^a \ln r_0 \, \delta_2^a e^{pq} - (1 \leftrightarrow 2) \tag{108}$$
$$= \{\delta_1^y \ln r_0 \, \delta_2^y e^{pq} - \delta_1^y \ln r_0 \, \delta_2^y \ln r_0 \partial_v e^{pq}\} - (1 \leftrightarrow 2) \tag{109}$$
$$= \delta_1^y \ln r_0 \, \delta_2^y e^{pq} - (1 \leftrightarrow 2). \tag{110}$$

Equation (107) can be demonstrated as follows: Any variation Δh_{tu} of the 2-metric on $\mathcal{N}_R$ gives rise to a variation

$$\Delta e^{pq} = \Delta[\sqrt{\det h} h^{pq}] = -h^{pr} h^{qs} \sqrt{\det h}[\Delta h_{rs} - \frac{1}{2} h_{rs} h^{tu} \Delta h_{tu}] \tag{111}$$

of the inverse conformal 2-metric. But because the v components of the induced metric on $\mathcal{N}_R$ vanish

$$\pounds_{\xi^v \partial_v} h_{rs} = \pounds_{\xi^v \partial_v} g_{rs} \tag{112}$$
$$= \xi^v \partial_v g_{rs} + \partial_r \xi^v g_{vs} + \partial_s \xi^v g_{rv} = \xi^v \partial_v h_{rs}. \tag{113}$$

The result (107) follows directly from this relation and (111). (See (124) for a general definition of the Lie derivative.)

Using the invariance of $\bar{\rho}$ the variation $\delta^y \ln r_0$ may be expressed in terms of $\delta^y \rho_0$. $r_0^2 = \rho_{y\,0}/\bar{\rho}$, where $\rho_{y\,0}$ is the area density on S_0 in the y chart, so, since $\delta^y \bar{\rho} = \delta^a \bar{\rho} = 0$,

$$\delta^y \ln r_0 = \frac{1}{2} \frac{\delta^y \rho_{y\,0}}{\rho_{y\,0}}. \tag{114}$$

But $\delta^y \rho_{y\,0}$ is just $\delta^y \rho_0$ transformed, as a density, from the θ chart to the y chart, so $\delta^y \rho_{y\,0}/\rho_{y\,0} = \delta^y \rho_0/\rho_0$. (See the discussion of the transformation under change of coordinates of comoving variations, such as $\delta^y \rho_0$, in appendix A.) Thus

$$\delta^y \ln r_0 = \frac{1}{2} \frac{\delta^y \rho_0}{\rho_0}. \tag{115}$$

The contribution (106) to ω_R can therefore be written as

$$\frac{1}{64\pi G} \int_{S_0} \delta_1^y \rho_0 \partial_v e_{pq} \delta_2^y e^{pq} \, d^2\theta - (1 \leftrightarrow 2). \tag{116}$$

Summing (104), (116), and (105) one obtains

$$\omega_R[\delta_1, \delta_2] = \frac{1}{16\pi G} \int_{S_0} d^2\theta \, \Bigg\{ \delta_1 \lambda \delta_2 \rho_0 + \delta_1 \tilde{\tau}_{R\,i} \delta_2 s_R^i + \frac{1}{4} \delta_1^y \rho_0 \partial_v e_{pq} \delta_2^y e^{pq} + \frac{1}{2} \rho_0 \int_1^{\bar{v}} v^2 \delta_1^a e^{pq} \partial_v \delta_2^a e_{pq} dv \Bigg\} - (1 \leftrightarrow 2), \tag{117}$$

The sum of (117) and its L branch analog is the desired expression for the symplectic 2-form $\omega_{\mathcal{N}} = \omega_R + \omega_L$ in terms of the free initial data, valid for

admissible variations. It coincides with the expression given in [Rei08] (although there $\delta^a e$ was called $\delta^\circ e$).

The variations appearing in (117) are not simply the variations of the components of the intial data fields. For instance, $\delta^a e_{pq}(\theta)$ is not the variation of $e_{pq}(\theta)$ but rather this variation minus $\pounds_\xi e_{pq}(\theta)$. Expressed directly in terms of the variations of the components of the initial data fields, each in a chart "natural" to it, the symplectic form is

$$\begin{aligned}\omega_R[\delta_1,\delta_2] = \frac{1}{16\pi G}\Big\{ & \tfrac{1}{2} \int_{S_R} d^2y\,\bar{\rho}\int_{r_0}^1 r^2 \delta_1 e^{ij}\,\partial_r \delta_2 e_{ij}\,dr \\ & +\tfrac{1}{4}\int_{S_0} \delta_1 \rho_{y\,0}\,\partial_v e_{ij}\,\delta_2[e^{ij}(y)]\,d^2y \\ & + \int_{S_0} [\delta_1 \lambda \delta_2 \rho_0 + \delta_1 \tilde{\tau}_{R\,i}\,\delta_2 s^i_R]\,d^2\theta - (1 \leftrightarrow 2)\Big\}. \quad (118)\end{aligned}$$

This is the expression given in [Rei07]. In the first, bulk, term the components of e are referred to the a chart; In the second term, a surface term, e and ρ_0 are referred to the y chart on S_0; And in the last term λ, ρ_0, $\tilde{\tau}_{R\,i}$, and s^i_R are referred to the θ chart. Of course the components s^i_R of s_R are also determined by the y chart. What is meant in this case is that the variation of these components is evaluated at constant θ.

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A Variations of fields and integrals

In the main text extensive use is made of charts that are adapted to the metric, in particular the a and b charts. We call such charts *moving charts* because they can change under variations of the fields. In the present appendix some basic facts about the variations of fields and their components in such charts are derived.

In our formalism fields will be defined by their components in charts. The components of a field F in a chart x, denoted $[F]_x$, is a collection of numbers, or more precisely, $\mathbb{C}$ valued functions of the coordinates x^μ of the chart x. Of the chart dependence of the components we will require only that within the domain of x the components of F in any other chart y are determined entirely by its x components and the transition function from the x to the y chart.

The advantage of this coordinate dependent representation of fields is that it allows us to treat all the fields that we encounter, tensors, densities, connection coefficients and others, in a uniform manner.

Recall that by virtue of its definition a manifold comes equiped with an atlas of charts (see for example [Wald84]). We will call these the *fixed* charts. A fixed chart assigns definite values to the coordinates at each manifold point in its domain. Moving charts are families of fixed charts depending on the values fields or parameters. The coordinate values they assign to points can depend on these fields or parameters. For instance, the a_A chart defined in subsection 4.3, depends on the metric and moves when the metric is varied.

The variation of a function f of a parameter λ when λ varies is simply another word for the derivative of f: $\delta f \equiv df/d\lambda$. We will be interested chiefly in the variations induced by variations of the metric field. Thus we have a family of metric fields parameterized by λ and we wish to find the derivative in λ of quantities calculated from the metric.

The variation of a function of λ only is thus unambigously defined. But a field depends also on position on the manifold, and its components depend on the chart used. Thus to define the variation of a field F one must define what it means to hold the position and the chart constant while λ is varied. Here we define δF in the usual way, as the λ derivative of F in a fixed chart. That is, we set the components $[\delta F]_x$ of δF in a fixed chart x equal to the variation of the x components of F at fixed values of the coordinates x:

$$[\delta F]_x = \delta[F]_x. \tag{119}$$

Variations may also be defined in an entirely analogous manner using moving charts: Let C be an atlas of comoving charts, that is, an atlas of charts which are λ dependent functions of the fixed charts, but have λ independent transition functions among themselves. Then the components of the *comoving variation* $\delta^C F$ in any chart $c \in C$ are

$$[\delta^C F]_c = \delta[F]_c, \tag{120}$$

where $\delta[F]_c$ denotes the variation of the c components at fixed values of the c coordinates. For any given value of λ the moving chart c coincides with a fixed chart c_λ, and the components of a field with respect to c are identified with the c_λ components of the field.[40] Note that within its domain a moving chart c defines its comoving atlas uniquely, and thus also the corresponding comoving variation, which may as well be written δ^c. Thus for instance the variation δ^{a_A} used in the main text is defined by the a_A chart.

Proposition:

$$\delta^C = \delta + \pounds_v. \tag{121}$$

Here $\pounds_v$ is the Lie derivative along the "velocity" v of the moving charts in C with respect to the fixed charts. v^μ is the λ rate of change of the fixed

[40] But $\delta[F]_c$ is the derivative $d/d\lambda$ of $[F]_c = [F]_{c_\lambda}$ holding constant the values of the c coordinates, not those of the c_λ coordinates at fixed λ.

chart coordinate x^μ corresponding to constant values of the coordinates of the moving charts in C. Equivalently, let $\Phi_{C\,\lambda}$ be the diffeomorphism such that $c^\alpha_\lambda(\Phi_{C\,\lambda}(p)) = c^\alpha_0(p)$ for each chart $c \in C$ and each point p in the domain of c_0. Then v is the velocity of the flow $\Phi_{C\,\lambda}$, i.e. the tangent of the curve $\lambda \mapsto \Phi_{C\,\lambda}(p)$ at $\lambda = 0$.

This proposition simply expresses the fact that the variation of $[F]_c$, the moving chart components of a field F, can be resolved into the sum of a variation, δ, holding the chart fixed, and a variation, $\pounds_v$, holding the fixed chart components of F fixed. Without loss of generality we may suppose that the variation is being evaluated at $\lambda = 0$. Then

$$[\delta^C F]_{c_0} \equiv \delta[F]_c = d/d\lambda([F]_{c_0}) + d/d\lambda([F_0]_c), \tag{122}$$

where F_0 is the field F at $\lambda = 0$ in the sense that $[F_0]_x = [F]_x$ at $\lambda = 0$ in any fixed chart x. The first term is $[\delta F]_{c_0}$. The second term turns out to be the Lie derivative of F.

The Lie derivative is defined in terms of the action of diffeomorphisms on the field (see [Wald84]). The action Φ^* of a diffeomorphism Φ on a field F satisfies the requierment that for any chart x

$$[\Phi^*(F)]_{\Phi^*(x)} = [F]_x, \tag{123}$$

where $\Phi^*(x) \circ \Phi = x$. That is, one requiers that if one acts on both the chart and the field with the same diffeomorphism then the components of the new field in the new chart are the same as those of the old field in the old chart. Thus $[F]_{c_\lambda} = [\Phi^{*\,-1}_{C\,\lambda}(F)]_{c_0}$ and thus at $\lambda = 0$

$$[\pounds_v F]_{c_0} \equiv -d/d\lambda[\Phi^*_{C\,\lambda}(F_0)]_{c_0} = d/d\lambda[\Phi^{*\,-1}_{C\,\lambda}(F_0)]_{c_0} = d/d\lambda[F_0]_{c_\lambda}, \tag{124}$$

which completes the proof of the proposition.

The variations δ^C may be interpreted in a different way, as variations with respect to the fixed charts that leave the moveable atlas C fixed. Indeed, since the variation δ in (121) determines the vector field v, δ^C may be regarded as a projection of δ to variations that fix C.[41]

How do the components of δF and $\delta^C F$ transform from one chart to another? Suppose x and y are two fixed charts. Recall that within the intersection of the domains of these charts the y components of a field F are determined by its x components via a transformation T depending only on the transition map $\varphi = y \circ x^{-1}$ between the charts themselves. If we assume that T is functionally differentiable in $[F]_x$ then

$$[\delta F]_y = \delta[F]_y = \delta T([F]_x) = DT \lrcorner\, \delta[F]_x, \tag{125}$$

where DT is the derivative of T and $\lrcorner$ denotes contraction. That is, δF transforms according to the linearization of the transformation of F. In the cases of

[41]It is important to remember that v depends on δ. Thus for instance, if δ already fixes C, then $v = 0$.

interest to us $[F]_y$ is a function only of $[F]_x$ at the same manifold point. That is, $[F]_y(y) = \tau([F]_x(\varphi^{-1}(y)))$ with $\varphi^{-1}(y)$ the x coordinates of the point defined by the values y of the y coordinates, and τ an ordinary function of the space of components at a point to itself. Then the transformation law for δF reduces to

$$[\delta F]_y = D\tau \lrcorner \, \delta[F]_x \circ \varphi^{-1}, \tag{126}$$

with D now the derivative in the space of components (which is finite dimensional for the fields we encounter) and the contraction also taken in this space.

The variations $\delta^C F$ transform in precisely the same way. Let x' and y' be the moving charts in C formed by carrying x and y along the flow of C: $x'_\lambda \circ \Phi_{C\,\lambda} = x$ and $y'_\lambda \circ \Phi_{C\,\lambda} = y$. Then, by (120), $[\delta^C F]_x = [\delta^C F]_{x'} = \delta[F]_{x'}$ and $[\delta^C F]_y = \delta[F]_{y'}$ at $\lambda = 0$. But the transition map from the x' to the y' chart is unaffected by the flow. It is just the transition map φ from x to y coordinates. The transformation from x' to y' components is thus also the same as that from x to y components: $[F]_{y'} = T([F]_{x'})$. It follows that $\delta^C F$ transforms according to

$$[\delta^C F]_y = \delta[F]_{y'} = \delta T([F]_{x'}) = DT \lrcorner \, \delta[F]_{x'} = DT \lrcorner \, [\delta^C F]_x, \tag{127}$$

just like δF.

The variations of several integrals are evaluated in the present work. In particular, part of the symplectic 2-form on $\mathcal{N}_R$ is obtained by varying the bulk term in the symplectic potential on $\mathcal{N}_R$. The latter is an integral of the form $\int_{\mathcal{N}_R} F$ where the integrand is represented by the 3-form F. Using the a_R chart this integral may be expressed as $\int_{y[S_0]} \int_{r_0}^1 [F]_a dr d^2 y$, an integral over a domain in $\mathbb{R}^3$. Its variation is therefore

$$\delta \int_{\mathcal{N}_R} F = \int_{y[S_0]} \int_{r_0}^1 \delta[F]_a dr d^2 y - \int_{y[S_0]} \delta[r_0]_y [F]_a|_{r=r_0} d^2 y. \tag{128}$$

Here $\delta[r_0]_y$ is the variation of the scalar r_0 at constant y. This variation may be expressed in a coordinate independent way in terms of the comoving variation δ^a associated with the a_R chart:

$$\delta \int_{\mathcal{N}_R} F = \int_{\mathcal{N}_R} \delta^a F - \int_{S_0} \delta[r_0]_y \partial_r \lrcorner \, F, \tag{129}$$

with S_0 oriented so that $\int_{S_0} dy^1 \wedge dy^2 = \int_{y[S_0]} d^2 y$.

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